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Superpotentials, A_∞ relations and WDVV equations for open topological strings

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Abstract: We give a systematic derivation of the consistency conditions which constrain open-closed disk amplitudes of topological strings. They include the A_∞ relations (which generalize associativity of the boundary product of topological field theory), as well as certain homotopy versions of bulk-boundary crossing symmetry and Cardy constraint. We discuss integrability of amplitudes with respect to bulk and boundary deformations, and write down the analogs of WDVV equations for the space-time superpotential. We also study the structure of these equations from a string field theory point of view. As an application, we determine the effective superpotential for certain families of D-branes in B-twisted topological minimal models, as a function of both closed and open string moduli. This provides an exact description of tachyon condensation in such models, which allows one to determine the truncation of the open string spectrum in a simple manner.

Keywords: Topological Field Theories, D-branes, Topological Strings.

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Contents

1. Introduction and summary

Closed topological string theories have played a very important role in the past [1]–[5]. They capture the exactly solvable sector of $N = 2$ supersymmetric string compactifications and govern holomorphic quantities such as the prepotential of the effective action. The tree-level amplitudes of topological strings can often be computed by geometric methods, see for example: [6]–[11]. These amplitudes satisfy a hierarchy of consistency conditions, such as crossing symmetry, factorization constraints, the WDVV equations [12] and $t - t^*$ equations [5], which are often strong enough to determine them.

At the level of (conformal) closed topological field theory (TFT) in two dimensions, one considers correlators of unintegrated and BRST closed zero-form observables. Such correlators must obey the sewing constraints of [13], namely crossing symmetry on the sphere and modular invariance on the torus. Restricting to tree-level, one is left with crossing symmetry only, which encodes associativity of the operator product. The WDVV equations are a consequence of this condition. They constrain tree level amplitudes, which are correlators on the sphere containing three zero-form observables and an arbitrary number of integrated two-form descendants. These relations can be derived within (twisted $N = 2$) TFT [12], and are part of a broader hierarchy of conditions known as the $t - t^*$ equations [5], which constrain topological string amplitudes at all genera.

The WDVV equations can be derived from the crossing symmetry relation as follows. One starts with the constraint:

$$\partial_i C_{jkl}(t) = \partial_j C_{ikl}(t) \,, \tag{1.1}$$

where

$$C_{ijk}(t) = \langle \phi_i \phi_j \phi_k e^{\sum_l t_\ell \int \phi_\ell^{(2)}} \rangle$$

are the triple correlators on the sphere deformed by integrated descendants:

$$\int \phi_\ell^{(2)} = \int dz \wedge d\bar{z} [G_{-1}, [\bar{G}_{-1}, \phi_\ell]] \,.$$

Here t_i are the flat deformation parameters and $\partial_i \equiv \partial/\partial t_i$. Equation (1.1) expresses the fact that the four-point function $C_{ijkl} \equiv \partial_i C_{jkl}$ is completely symmetric in all indices. This implies integrability of the deformed triple correlator:

$$C_{ijk}(t) = \partial_i \partial_j \partial_k \, \mathcal{F}(t) \,,$$

where $\mathcal{F}$ is the generating function of genus zero amplitudes, known as the WDVV potential. In appropriate situations, this quantity can be interpreted as the effective prepotential of a Calabi-Yau compactification of the associated untwisted superstring theory.

The crossing symmetry of four-point correlators now gives a system of equations for $\mathcal{F}$:

$$\partial_i \partial_k \partial_m \mathcal{F} \, \eta^{mn} \, \partial_n \partial_j \partial_l \mathcal{F} = \partial_i \partial_j \partial_m \mathcal{F} \, \eta^{mn} \, \partial_n \partial_k \partial_l \mathcal{F} \,,$$

where η^{mn} denotes the inverse of the topological metric $\eta_{mn} = C_{0mn}$, which one can show to be independent of the deformation parameters t_i. These are the famous associativity,

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or WDVV equations [12]. They constrain the generating function $\mathcal{F}(t)$, and encode a Frobenius structure on the associated moduli space [14].

In principle, the WDVV equations allow one to determine $\mathcal{F}$ without directly computing all genus zero amplitudes.[1] Such amplitudes are affected by contact terms arising from colliding operators, and therefore are hard to compute directly. The $t-t^*$ extension of the WDVV system constrains higher genus contributions to the generating function, and can be used to determine the gravitational F-terms of the corresponding effective action [5].

For open topological strings, which correspond to world-sheets with boundaries, the situation is much more complicated. Although geometric methods have been successfully applied to computing non-trivial effective $N = 1$ superpotentials from D-branes (see e.g., [18]–[25]), they were tailored to specific D-brane geometries (notably non-compact toric Calabi-Yau manifolds and their mirrors) and it is not obvious how these methods can be generalized to other classes of D-brane backgrounds.

Experience with closed string TFT suggests that a good strategy should be to first study the consistency conditions constraining open-closed string amplitudes, and then translate the results into a geometric structure associated with the moduli space of D-branes. The work of [26] (see also [27, 28]) discussed particular cases of the relevant algebraic constraints for open-closed topological strings (the basic conditions on pure boundary amplitudes on the disk were given in [29] in the context of open string field theory). However, the full set of constraints on open-closed amplitudes on the disk has not yet been worked out completely, and in particular was not clearly understood for deformation families of amplitudes with integrated insertions.

A new aspect of open topological strings is the generic lack of integrability of disk amplitudes with respect to boundary deformation parameters, and relatedly, the lack of flat coordinates. This arises because such amplitudes are only cyclically symmetric with respect to boundary insertions. Another complication is the fact that open topological field theories in two dimensions have more sewing constraints than their closed counterparts. As shown in [30]–[33], one finds four sewing conditions involving boundary amplitudes, namely crossing symmetry on the disk, two bulk-boundary crossing relations and a topological version of the Cardy constraint (the other two conditions involve only bulk correlators). As a consequence, one has more families of algebraic and differential conditions on deformed tree-level open-closed amplitudes.

The first constraint mentioned above encodes associativity of the boundary operator product. Because the product fails to be commutative, the correct stringy generalization of this condition turns out to be more complicated than for closed strings. Namely, one finds a series of equations reflecting an A_∞ structure, which can be derived by using the Ward identities of the BRST operator. This means that the associated 'string products' (the stringy generalizations of the boundary operator product) are associative 'up to homotopies'. This A_∞ structure is cyclic, due to cyclicity of disk amplitudes in the boundary insertions; it is also unital and minimal. The fact that tree-level open string products obey A_∞ constraints was originally pointed out in [29] in the context of open string field theory

[1] Some applications along these lines can be found in [12, 15, 16, 14, 17].

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(see also [34]). It was further discussed in [35, 36, 37] as the underlying structure controlling D-brane superpotentials, following ideas originally put forward in [38]. Further discussion of such relations in the context of bosonic string field theory can be found in [39, 40]. Such constraints also played a role in [26, 28, 41]. A_∞ constraints are central to the homological mirror symmetry program [42]–[49], where they arise via open string field theory [50] (see also [51]–[55]).

Upon perturbing boundary disk amplitudes via bulk insertions, the A_∞ algebra deforms in a manner compatible with cyclicity. Since linearized deformations of cyclic A_∞ algebras are controlled by their cyclic complex, the first order approximation leads to a map which associates a cocycle of this complex to each BRST-closed bulk insertion. In similar manner, the appropriate generalization of the remaining sewing constraints (namely bulk-boundary crossing symmetry and the Cardy condition) is given by two countable sets of algebraic conditions on bulk-boundary amplitudes on the disk.

Our main purpose is to derive this series of constraints from the Ward identities of a general, twisted $N = 2$ topological theory on the disk. We shall express these conditions as a countable family of nonlinear algebraic and differential equations which constrain the moduli-dependent disk amplitudes; these relations constitute the open string analogs of the WDVV equations.

The paper is organized as follows. In section 2, we briefly recall some basic features of open-closed topological field theories. In section 3, we define the genus zero, deformed open string amplitudes:

$$B_{a_0\dots a_m;i_1\dots i_n} \;=\; (-1)^{\tilde a_1+\dots+\tilde a_{m-1}}\langle\, \psi_{a_0}\psi_{a_1}\, P\int \psi^{(1)}_{a_2}\dots\int \psi^{(1)}_{a_{m-1}}\,\psi_{a_m}\int \phi^{(2)}_{i_1}\dots\int \phi^{(2)}_{i_n}\,\rangle_{\rm disk}\,,$$

and discuss their basic properties such as independence of the worldsheet metric, cyclicity with respect to boundary insertions and constancy of the boundary two-point function. A shift in the grading of the boundary fields, denoted by $\tilde a_i \equiv a_i + 1 \bmod 2$, will play a crucial role. This shift reflects the change in the degree of operators induced by super-integration over the moduli of boundary insertions, and gives a physical realization of the "suspension" operation used in the mathematics literature.

In section 4, we discuss the issue of integrability with respect to the bulk and boundary perturbation parameters, denoted by t_i and s_a, respectively. We show that correlators deformed by bulk operators integrate to disk amplitudes $\mathcal{F}_{a_0\dots a_m}(t)$, so that:

$$B_{a_0\dots a_m;i_1\dots i_n} = \partial_{i_1}\dots\partial_{i_n}\mathcal{F}_{a_0\dots a_m}(t)|_{t=0}\,. \tag{1.6}$$

Because disk amplitudes are only cyclically (rather than completely) symmetric in the boundary fields, they are *a priori* not integrable with respect to the boundary deformation parameters. However, by promoting these to formal non-commutative variables $\hat s_a$, one can define a formal generating function $\hat{\mathcal{W}}(\hat s, t)$ through the expansion:

$$\hat{\mathcal{W}}(\hat s, t) = \sum_{m\geq 1}\frac{1}{m}\hat s_{a_m}\dots\hat s_{a_1}\mathcal{F}_{a_1\dots a_m}(t)\,. \tag{1.7}$$

This generating function encodes all information contained in the disk correlators, and can be used to define a sort of formal noncommutative Frobenius (super)manifold. To recover a physically meaningful quantity, one can impose (super)commutativity of $\hat{s}_a$ by working modulo the ring generated by their commutators. Denoting the resulting equivalence classes by s_a, this gives a quantity $\mathcal{W}(s,t)$ which in the appropriate framework can be identified with the effective $N=1$ superpotential of a four-dimensional superstring compactification with D-branes, where s_a and t_i correspond to vacuum expectation values.

Notice the difference to the bulk effective prepotential $\mathcal{F}(t)$: its derivatives directly yield all bulk topological amplitudes, while the s_a-derivatives of $\mathcal{W}(s,t)$ give only sums of correlators, namely the (super)symmetrized version of the boundary amplitudes:

$$\mathcal{A}_{a_0\ldots a_m}(t) \equiv m!\,\mathcal{F}_{(a_0\ldots a_m)}(t)\,.$$

The effective superpotential $\mathcal{W}(s,t)$ thus contains *less* information than that provided by the full collection of disk amplitudes. This is exemplified by the consistency relations when applied to the super-symmetrized quantities $\mathcal{A}_{a_0\ldots a_m}(t)$: sometimes these equations are nearly empty, in contrast to the full constraints on the original amplitudes $\mathcal{F}_{a_0\ldots a_m}(t)$, which are only cyclically symmetric. The conditions on $\mathcal{W}(s,t)$ induced by the consistency conditions can be viewed as a weak form of the generalized WDVV equations.

In section 5, we begin our analysis of the Ward identities, by first deriving a series of conditions on undeformed boundary amplitudes on the disk, which generalize associativity of the boundary product. These take the form:

$$\sum(-1)^{\tilde{a}_1+\ldots+\tilde{a}_{l-2}} B^b{}_{a_1\ldots a_{l-2}ca_{k+1}\ldots a_m} B^c{}_{a_{l-1}\ldots a_k} = 0\,,$$

and encode a so-called minimal A_∞ structure. Due to cyclicity of amplitudes with respect to boundary insertions, this is in fact a cyclic A_∞ algebra; we also show that it admits a unit.

Section 6 considers general bulk-boundary amplitudes on the disk. We discuss deformations induced by an arbitrary number of bulk insertions, and show that the deformed boundary amplitudes $\mathcal{F}_{a_0\ldots a_m}(t)$ preserve a weak, unital and cyclic A_∞ structure. In Subsection 6.2, we extract certain identities generalizing the bulk-boundary crossing symmetry constraint of two-dimensional TFT. These identities take the form:

$$\begin{aligned}\partial_i\partial_j\partial_k\mathcal{F}(t)\ \eta^{kl}\ \partial_l\mathcal{F}_{a_0a_1\ldots a_m}(t) \;=\; &\sum(-1)^{\tilde{a}_{m_1+1}+\ldots+\tilde{a}_{m_3}}\mathcal{F}_{a_0\ldots a_{m_1}ba_{m_2+1}\ldots a_{m_3}ca_{m_4+1}\ldots a_m}(t)\,\times\\ &\times\partial_i\mathcal{F}^b{}_{a_{m_1+1}\ldots a_{m_2}}(t)\partial_j\mathcal{F}^c{}_{a_{m_3+1}\ldots a_{m_4}}(t)\,.\end{aligned}$$

Subsection 6.3 discusses the generalization of the topological Cardy constraint, which is given by the following series of equations:

$$\begin{aligned}\partial_i\mathcal{F}_{a_0\ldots a_n}(t)\eta^{ij}\partial_j\mathcal{F}_{b_0\ldots b_m}(t) \;=&\sum(-1)^{s+\tilde{c}_1+\tilde{c}_2}\,\omega^{c_1d_1}\,\omega^{c_2d_2}\,\mathcal{F}_{a_0\ldots a_{n_1}d_1b_{m_1+1}\ldots b_{m_2}c_2a_{n_2+1}\ldots a_n}(t)\times\\ &\times\mathcal{F}_{b_0\ldots b_{m_1}c_1a_{n_1+1}\ldots a_{n_2}d_2b_{m_2+1}\ldots b_m}(t)\,.\end{aligned}$$

In section 7, we demonstrate the power of the open string consistency conditions by applying them to topological minimal models with a boundary. We find that they give a highly overdetermined system of equations which uniquely determines all disk amplitudes (as functions of both open and closed string deformation parameters) thereby fixing the effective superpotential $\mathcal{W}(s,t)$. As a consistency check, we show that the critical loci of $\mathcal{W}(s,t)$ correspond to a factorized Landau-Ginzburg superpotential, which is known [56]–[61] to be the criterion for unbroken supersymmetry. Starting from the unperturbed theory and moving along these loci describes an exactly solvable, topological version of tachyon condensation, which truncates the open string spectrum in a computable manner.

2. Preliminaries

2.1 Bulk topological conformal field theory

Let us start with a brief review of bulk topological conformal field theory [12, 5]. This will prove useful later on, when we discuss the boundary extension. We shall consider a closed conformal field theory with topological conformal symmetry on the worldsheet. Denoting by $T(z)$ and $\bar{T}(\bar{z})$ the left and right moving components of the stress-energy tensor, this implies the existence of odd scalar charges $Q_0, \bar{Q}_0$ and spin two fermionic currents $G(z)$ and $\bar{G}(\bar{z})$ such that $Q_0^2 = \bar{Q}_0^2 = 0$ and:

$$T(z) = [Q_0, G(z)]\,, \tag{2.1}$$

with a similar relations for the right movers. We also have U(1) charges J and $\bar{J}$ with the property:

$$[J, T(z)] = 0\,, \qquad [J, Q_0] = Q_0\,, \qquad [J, G(z)] = -G(z)\,,$$

and a similar relation for the left movers. Here and below we use $[\cdot,\cdot]$ to denote the *super*commutator. This data defines what is generally called a *string background* [62, 63].

Using the mode expansions:

$$\begin{aligned} T(z) &= \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \\ G(z) &= \sum_{n\in\mathbb{Z}} G_n z^{-n-2}\,, \end{aligned}$$

one finds the algebra:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n}\,, & [L_m, G_n] &= (m-n)G_{m+n} \\ [L_m, Q_0] &= 0\,, & [L_m, J_0] &= 0 \\ [G_m, Q_0] &= L_m\,, & [J_0, J_0] &= 0 \\ [J_0, G_n] &= G_n\,, & [J_0, Q_0] &= -Q_0\,, \end{aligned} \tag{2.3}$$

with similar relations constraining the right movers.

We let $\phi_i(z,\bar{z})$ $(i = 0 \ldots h_c - 1)$ be a collection of zero-form operators which satisfy:

$$[Q_0, \phi_i] = [\bar{Q}_0, \phi_i] = 0\,.$$

We shall assume that this system is complete in the sense that it descends to a basis of the space of on-shell observables[2] H_c, which is the double BRST cohomology computed

[2] Through the operator-state correspondence, H_c can be identified with the space of on-shell oscillation states of the closed topological string. Then ϕ_i can be viewed as linear operators on H_c.

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with Q_0 and $\bar{Q}_0$. We choose ϕ_i such that ϕ_0 coincides with the bulk identity operator 1_c. For simplicity, we shall also assume that each ϕ_i is Grassmann even. This simplifies certain sign prefactors in later sections and suffices for our main application, which concerns topological Landau-Ginzburg models. With this assumption, H_c is a complex vector space of dimension h_c (as opposed to a super-vector space, which is the general case).

Given such operators, one can construct their descendants by using relation (2.1) and its counterpart for the right movers, which imply:

$$[Q_0, G_{-1}] = L_{-1}\,, \qquad [\bar{Q}_0, \bar{G}_{-1}] = \bar{L}_{-1}\,.$$

Since the commutator with L_{-1} and $\bar{L}_{-1}$ acts as $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$, we find that the operators:

$$\begin{aligned}
\phi_i^{(1,0)} &= [G_{-1}, \phi_i]dz \\
\phi_i^{(0,1)} &= [\bar{G}_{-1}, \phi_i]d\bar{z} \\
\phi_i^{(1,1)} &= [G_{-1}, [\bar{G}_{-1}, \phi_i]]dz \wedge d\bar{z} = [\bar{G}_{-1}, [G_{-1}, \phi_i]]d\bar{z} \wedge dz
\end{aligned}$$

satisfy the descent equations:

$$\begin{aligned}
&[Q_0, \phi_i^{(1,0)}] = \partial\phi_i\,, && [\bar{Q}_0, \phi_i^{(1,0)}] = 0 \\
&[Q_0, \phi_i^{(0,1)}] = 0\,, && [\bar{Q}_0, \phi_i^{(0,1)}] = \bar{\partial}\phi_i \\
&[Q_0, \phi_i^{(1,1)}] = \partial\phi_i^{(0,1)} = d\phi_i^{(0,1)}\,, && [\bar{Q}_0, \phi_i^{(1,1)}] = \bar{\partial}\phi_i^{(1,0)} = d\phi_i^{(1,0)}\,.
\end{aligned} \tag{2.6}$$

Notice that $\phi_i^{(1,0)}$ and $\phi_i^{(0,1)}$ are operator-valued sections of the canonical and anticanonical line bundles over $\mathbb{P}^1$, while $\phi_i^{(1,1)}$ is an operator-valued two-form. The integrated operators:

$$\int_{S^2} \phi_i^{(1,1)} \tag{2.7}$$

are both Q_0- and $\bar{Q}_0$-closed. One can also write down BRST-closed loop integrals of the one-form descendants $\phi_i^{(0,1)} + \phi_i^{(1,0)}$, but those observables do not play a role for what follows.

The fundamental correlators of the topological field theory at tree level are the two- and tree-point functions $\eta_{ij} := \langle\phi_i\phi_j\rangle$ and $C_{ijk} = \langle\phi_i\phi_j\phi_k\rangle$ of zero-form observable. Notice that $\eta_{ij} = C_{0ij}$. It is known that this quantity defines a non-degenerate symmetric pairing on the space of zero-form observables. Any tree-level correlator of the form $\langle\phi_{i_1}\ldots\phi_{i_n}\rangle$ depends only on the BRST cohomology classes of ϕ_{i_j}, is completely symmetric under permutations of these operators, and factorizes as a sum of products of three-point functions, with insertions of the inverse η^{ij} of the topological metric. At tree level, compatibility of various factorizations of a given n-point function is assured by a single sewing constraint, namely crossing symmetry of the four point function on the sphere [13]. This condition says that the three distinct factorizations of this function must agree:

$$\langle\phi_i\phi_j\phi_k\phi_l\rangle = C_{ijm}\eta^{mn}C_{nkl} = C_{ikm}\eta^{mn}C_{njl} = C_{ilm}\eta^{mn}C_{njk}\,. \tag{2.8}$$

Due to symmetry of C_{ijk} under permutations of its indices, only one of the two equalities to the right is an independent condition.

We next consider tree-level amplitudes,[3] which have the form:

$$C_{i_1\dots i_n} := \langle \phi_{i_1}\dots\phi_{i_3}\int\phi^{(1,1)}_{i_4}\dots\int\phi^{(1,1)}_{i_n}\rangle_{S^2}\,. \qquad (2.9)$$

These differ from topological field theory correlators since they contain integrated insertions of higher form operators. Such insertions implement integration of the underlying zero-form correlator $\langle\phi_{i_1}\dots\phi_{i_n}\rangle_{S^2}$ over an appropriate compactification of the configuration space of n points on the sphere.

To derive the WDVV equations, one uses conformal invariance and the Ward identities for the supercurrents $G(z)$ and $\bar{G}(\bar{z})$. Though in this paper we assume full twisted $N=2$ topological symmetry, it is worth noting that the derivation of [12] only depends on the following properties:

(A) Q_0 and $\bar{Q}_0$ are symmetries of the theory

(B) G_{-1}, G_0, G_1 and their right-moving counterparts are symmetries.

Through equations (2.1), this implies that the $\mathrm{PSL}(2,\mathbb{C})$ group generated by L_{-1}, L_0 and L_1 and their right-moving counterparts is also a symmetry. Using these assumptions, one can show that [12]:

(I) The tree-level string amplitudes $C_{0i_2\dots i_n}$ vanish for $n\geq 3$.

(II) The amplitudes $C_{i_1\dots i_n}$ are symmetric under permutations of $i_1\dots i_n$.

Let us define perturbed string amplitudes by the expression:

$$C_{i_1\dots i_n}(t) = \left\langle \phi_{i_1}\phi_{i_2}\phi_{i_3}\int_{S^2}\phi^{(1,1)}_{i_4}\dots\int_{S^2}\phi^{(1,1)}_{i_n}e^{\sum_{p=0}^{h_c-1}t_p\int_{S^2}\phi_p^{(1,1)}}\right\rangle,$$

which is understood as the formal power series:

$$C_{i_1\dots i_n}(t) = \sum_{N_0\dots N_{h_c-1}=0}^{\infty}\prod_{p=0}^{h_c-1}\frac{t_p^{N_p}}{N_p!}\left\langle\phi_{i_1}\phi_{i_2}\phi_{i_3}\int_{S^2}\phi^{(1,1)}_{i_4}\dots\int_{S^2}\phi^{(1,1)}_{i_n}\prod_{p=0}^{h_c-1}\left[\int_{S^2}\phi_p^{(1,1)}\right]^{N_p}\right\rangle.$$

Here $t=(t_0\dots t_{h_c-1})$ is a collection of complex-valued parameters. Using property (II), we can express all deformed amplitudes on the sphere with at least four insertions as partial derivatives of the deformed three-point function:

$$C_{i_1\dots i_n}(t) = \partial_{i_4}\dots\partial_{i_n}C_{i_1i_2i_3}(t)|_{t=0} \qquad \text{for } n\geq\ .$$

Here and below we use the notation $\partial_i := \partial/\partial t_i$.

[3]Throughout this paper, we shall use the term *amplitudes* for correlators containing integrated insertions of descendants. Correspondingly, correlators without integrated descendants will be referred to as *TFT correlators*. With this distinction, amplitudes can be represented as integrals of correlators over the moduli space of the underlying Riemann surface with punctures. Notice that TFT correlators can be described entirely in the simple algebraic framework of topological field theory, which reduces them to algebraic building blocks (see [31, 33, 32] for a complete analysis in the open-closed case). On the other hand, amplitudes are related to scattering in the topological string theory built by considering such a TFT on the worldsheet. It is such amplitudes which form the main focus of the present paper.

Then property (I) shows that the perturbed topological metric $\eta_{ij}(t) := C_{0ij}(t)$ is independent of the parameters t. On the other hand, property (II) implies that $C_{i_1 \ldots i_n}(t)$ are symmetric under all permutations of indices. In particular, we find that $\partial_i C_{jkl}(t)$ is completely symmetric in i, j, k and l. This is an integrability property allowing us to write the deformed three-point correlator as a triple derivative of a function $\mathcal{F}(t)$:

$$C_{ijk}(t) = \partial_i \partial_j \partial_k \mathcal{F}(t)\,. \tag{2.11}$$

The generating function $\mathcal{F}$ is known as the WDVV potential. In the appropriate geometric set-up, it can be interpreted as the prepotential of the effective space-time theory associated with an $N = 2$ Calabi-Yau compactification of a superstring model (to which the topological worldsheet theory is related by twisting).

The WDVV equations [12] are the conditions that the quantities η_{ij} and $C_{ijk}(t)$ can be viewed as the two and three point functions of a deformed topological field theory. Since we work at tree-level, this amounts to the requirement that the deformed three-point functions satisfy the sewing constraint (2.8) *for finite* t :

$$C_{ijm}(t)\eta^{mn}C_{nkl}(t) = C_{ikm}(t)\eta^{mn}C_{njl}(t)\,. \tag{2.12}$$

Using relation (2.11), this gives a system of second order, quadratic partial differential equations for the prepotential:

$$\partial_i \partial_j \partial_m \mathcal{F}\ \eta^{mn}\ \partial_n \partial_k \partial_l \mathcal{F} = \partial_i \partial_k \partial_m \mathcal{F}\ \eta^{mn}\ \partial_n \partial_j \partial_l \mathcal{F}\,. \tag{2.13}$$

These are the well-known associativity, or WDVV relations [12].

Remark. The sewing constraints (2.12) can be viewed as integrability conditions for the existence of a deformed topological field theory at finite t. They must be satisfied if deforming the worldsheet action by the infinitesimal term:

$$\delta S = \sum_{i=0}^{h_c - 1} t_i \int \phi_i^{(1,1)} \tag{2.14}$$

is to lead to a quantum worldsheet theory which satisfies the (tree-level) topological sewing constraint, when such deformations are extended to finite t. One case in which this is guaranteed is for those perturbations which are *exactly* marginal and preserve the symmetries Q, G_{-1}, G_0 and G_1 as well as their right-moving counterparts, or a deformation thereof which satisfy the same algebra. In this case, relation (2.1) and its right-moving counterpart continue to hold in the deformed theory, which therefore is topological. Since the variation (2.14) is BRST closed but not exact, the topological character of the theory cannot generally be preserved without modifying the BRST operator. Similarly, the generators Q, G_{-1}, G_0 and G_1 must change in a t-dependent manner. As a consequence, the descendants $\phi_i^{(1,1)}$ will also depend on t (though their cohomology classes need not), and extending (2.14) to finite t becomes nontrivial. This generally makes it difficult to construct appropriate deformations of the worldsheet theory. The WDVV equations (2.13) are the minimal (tree-level) requirement for such deformations to exist. In applications, it is often nontrivial to build appropriate deformations of the worldsheet model, even in those situations where the WDVV potential can be determined independently from equations (2.13). A well-known example is the topological B-model of [3], for which the full WDVV potential (including the part depending on odd parameters t) was constructed in [64] and the underlying deformation of the worldsheet theory was given only relatively recently in [65].

2.2 Adding a boundary

Extending the analysis of TCFT to worldsheets with boundaries induces several profound changes in the structure of the theory. An important new aspect concerns the integrated operators (2.7), which fail to be Q-closed due to the presence of the boundary. We shall return to this point later, after studying the boundary conditions. As above, we concentrate on tree-level amplitudes, which in this context are the (integrated) correlation functions on the disk, or equivalently the upper half-plane. We let $z = \tau + i\sigma$ with τ, σ the real coordinates of the complex plane.

We shall require that the boundary conditions preserve the transformations generated by $Q := Q_0 + \bar{Q}_0$ and $J = J_0 + \bar{J}_0$ and that the following condition holds at the boundary:

$$G(z) = \bar{G}(\bar{z}) \qquad \text{for } z = \bar{z}\,.$$

Due to equation (2.1), these constraints ensure that there is no flow of energy across the boundary, i.e. we have $T(z) = \bar{T}(\bar{z})$ for $z = \bar{z}$. Let us define:

$$Q := Q_0 + \bar{Q}_0\,, \qquad G := G_{-1} + \bar{G}_{-1}\,.$$

Then $Q^2 = 0$ and the following identity holds:

$$[Q, G] = L_{-1} + \bar{L}_{-1}\,. \tag{2.15}$$

Also notice that the transformations generated by Q and $G, G_0 + \bar{G}_0, G_1 + \bar{G}_1$ as well as $L_{-1} + \bar{L}_{-1}, L_0 + \bar{L}_0, L_1 + \bar{L}_1$ are symmetries of the theory defined on the disk. This follows from our boundary conditions and from properties (A) and (B) discussed in the previous subsection. The last three operators generate the group $\mathrm{PSL}(2,\mathbb{R})$ of global conformal symmetries of the disk.

2.2.1 Zero-form observables and sewing constraints

When constructing observables, we must consider both bulk zero-form operators ϕ_i and zero-form operators ψ_a $(a = 0 \ldots h_o - 1)$ supported on the boundary. The bulk zero-forms are as before. For the boundary operators, we assume $[Q, \psi_a] = 0$ and choose a collection which induces a basis of the Q-cohomology H_o of boundary operator-valued zero-forms. We shall allow the space H_o to carry a $\mathbb{Z}_2$-grading denoted by $|\cdot|$.[4] This complex super-vector

[4]In general, this is the sum of the Grassmann degree and another $\mathbb{Z}_2$ grading arising from a 'brane-antibrane' structure in the boundary sector. The later occurs by describing boundary data through a topological version of tachyon condensation between 'elementary D-branes'. Examples are discussed in [52, 54, 53, 55] for topological sigma models and in [58, 60, 61] for topological Landau-Ginzburg models. We stress that such a grading on the space of boundary observables seems to be essential in almost any realistic model, at least if one wishes to consider reasonably general D-branes. As a consequence, the boundary sector of the worldsheet topological field theory must be described in the $\mathbb{Z}_2$-graded framework discussed in [33], which generalizes the analysis performed in [31, 32] for the ungraded case.

space of dimension h_o becomes an associative superalgebra over the complex numbers when endowed with the composition given by the boundary product:

$$(\psi_a, \psi_b) \to \psi_a \psi_b = D_{ab}^c \psi_c \,, \tag{2.16}$$

where $D_{ab}^c \in \mathbb{C}$ are the structure constants of the boundary OPE. Associativity follows from one of the sewing constraints, namely boundary crossing symmetry:

$$D_{ab}^d D_{dc}^e = D_{ad}^e D_{bc}^d \,. \tag{2.17}$$

In particular, we notice compatibility between the grading and boundary product:

$$|\psi_a \psi_b| = |\psi_a| + |\psi_b| \,.$$

For ease of notation, we shall also denote the degree of ψ_a by $|a|$:

$$|a| := |\psi_a| \in \mathbb{Z}_2 \,. \tag{2.19}$$

The boundary two-point function on the disk:

$$\omega(\psi_a, \psi_b) := \langle \psi_a \psi_b \rangle := \omega_{ab} \tag{2.20}$$

defines a non-degenerate bilinear form on H_o, which satisfies the graded symmetry property:

$$\omega_{ab} = (-1)^{|a||b|} \omega_{ba} \tag{2.21}$$

and the selection rule:

$$\omega_{ab} = 0 \qquad \text{unless } |a| + |b| = |\omega| \mod 2 \,, \tag{2.22}$$

where $|\omega| \in \mathbb{Z}_2$ is a model-dependent degree. This bilinear form is known as the *boundary topological metric* [32, 33]. We shall denote its inverse by ω^{ab}.

One has the compatibility property:

$$\omega(\psi_a, \psi_b \psi_c) = \omega(\psi_a \psi_b, \psi_c) \,,$$

which amounts to cyclicity of ω when combined with (2.21):

$$\omega(\psi_a, \psi_b \psi_c) = (-1)^{|c|(|a|+|b|)} \omega(\psi_c, \psi_a \psi_b) \,.$$

Defining $D_{abc} := \omega_{ae} D_{bc}^e$, the last relation becomes:

$$D_{abc} = (-1)^{|c|(|a|+|b|)} D_{cab} \,.$$

As explained in [33], the boundary algebra admits a unit $1_o \in H_o$ (which automatically has even degree). We shall chose ψ_a such that $\psi_0 := 1_o$. With this choice, we have:

$$D_{0b}^a = D_{b0}^a = \delta_b^a \,.$$

Other important data are the bulk-boundary two-point function on the disk, which can be related to the boundary and bulk topological metrics with the help of the so-called bulk-boundary and boundary-bulk maps[5] of [33]:

$$\langle\phi_i\psi_a\rangle_{\rm disk} = \omega(e(\phi_i),\psi_a) = \eta(\phi_i, f(\psi_a))\,.$$

Writing $e(\psi_i) = e_i^a\psi_a$ and $f(\psi_a) = f_a^i\phi_i$, we find the adjunction relation:

$$e_{ia} = f_{ai}\,,$$

where:

$$e_{ia} := e_i^b\omega_{ba} \quad , \quad f_{ai} := \eta_{ij}f_a^j\,.$$

One has $e(1_c) = 1_o$, i.e:

$$e_0^a = \delta_0^a\,.$$

As discussed in [33], the data (D,e) is subject to the constraints:

$$D_{ab}^c e_i^b = e_i^b D_{ba}^c \tag{2.31}$$

$$D_{ab}^c e_i^a e_j^b = C_{ij}^k e_k^c\,, \tag{2.32}$$

which encode the two basic bulk-boundary sewing conditions. These relations mean that H_o becomes a (unital) associative superalgebra over the bulk ring H_c if the external multiplication is defined through:

$$\phi_i\psi_a := e(\phi_i)\psi_c = e_i^b C_{ba}^c\psi_c\,.$$

The remaining sewing constraint involving boundary data is the Cardy condition, which can be expressed as follows:

$$\eta_{ij}e_{ia}e_{jb} = (-1)^{s(c,d)}D_{ad}^c D_{cb}^d\,, \tag{2.34}$$

where $s(c,d)$ is a model-dependent sign and summation over d is understood.

2.2.2 Descendants

In the presence of the boundary, the descent equations for bulk operators must be taken with respect to the BRST charge $Q = Q_0 + \bar{Q}_0$. In particular, we shall consider a single one-form descendant for each ϕ_i. Adapting the construction of the previous section, we define:

$$\phi_i^{(1)} = \phi_i^{(1,0)} + \phi_i^{(0,1)}$$

$$\phi_i^{(2)} = \phi_i^{(1,1)}\,.$$

Using equations (2.6), we find the descent relations:

$$[Q,\phi_i^{(1)}] = d\phi_i$$

[5]The bulk-boundary map e is a map from H_c to H_o, while the boundary-bulk map f is a map from H_o to H_c. Both maps are complex-linear.

$$[Q, \phi_i^{(2)}] = d\phi_i^{(1)} . \tag{2.35}$$

The last of these equations implies:

$$[Q, \int_{D^2} \phi_i^{(2)}] = \int_{\partial D^2} \phi_i^{(1)} . \tag{2.36}$$

Notice the presence of a boundary term on the right-hand side. Defining:

$$\psi_a^{(1)} := [G, \psi_a] d\tau$$

and using relation (2.15), we also find the boundary descent equation:

$$[Q, \psi_a^{(1)}] = \left(\frac{d}{d\tau}\psi_a\right) d\tau . \tag{2.38}$$

Since operators will be inserted on the boundary in cyclic order, the typical integral of a descendant $\psi_a^{(1)}$ runs from the insertion to its left to the insertion to its right:

$$\int_{\tau_l}^{\tau_r} \psi_a^{(1)} . \tag{2.39}$$

Here 'left' and 'right' should be understood in the sense of the cyclic order on the boundary of the disk, which is determined by the orientation on the boundary induced from the orientation of the interior. As a consequence, we find that the BRST variation of (2.39) need not vanish:

$$\left[Q, \int_{\tau_l}^{\tau_r} \psi_a^{(1)}\right] = \psi_a \Big|_{\tau_l}^{\tau_r} . \tag{2.40}$$

Notice that the Grassmann degree of $\psi_a^{(1)}$ is opposite to that of ψ_a. It is convenient to take this into account by introducing a new grading on the boundary algebra H_o:

$$\deg \psi = |\psi| + 1 \pmod 2 .$$

For ease of notation, this shifted, or "suspended" grade of ψ_a will be denoted by a tilde:

$$\tilde{a} := \deg\, \psi_a = |a| + 1 \pmod 2 . \tag{2.42}$$

In terms of the suspended grade, the selection rule (2.22) becomes:

$$\omega_{ab} = 0 \qquad \text{unless } \tilde{a} + \tilde{b} = \tilde{\omega} + 1 \pmod 2 , \tag{2.43}$$

where $\tilde{\omega} = |\omega| + 1 \in \mathbb{Z}_2$. Moreover, the graded symmetry property (2.21) of the boundary 2-point function takes the form:

$$\omega_{ab} = (-1)^{\tilde{\omega}} (-1)^{\tilde{a}\tilde{b}} \omega_{ba} . \tag{2.44}$$

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Remark. In string theory we have in addition to open strings attached to a single D-brane (or a stack of D-branes), also strings which are stretched between two different D-branes. It is well-known [66] that the former correspond to boundary preserving operators ψ_a^{AA}, whereas the latter correspond to boundary condition changing operators ψ_a^{AB} (since they mediate between two different boundary conditions (D-branes) labeled by A and B). Of course, all operators of the topological conformal algebra (2.3) are boundary preserving, since they are related by a single condition on the boundary. The action of the charges on boundary condition changing operators can be written in a form which is very similar to that relevant for the boundary preserving sector. For example:

$$[G, \psi_a]^{AB} = G^{AA}\ \psi_a^{AB} - (-1)^{|a|}\psi_a^{AB}\ G^{BB} := \oint (G(z)\ \psi_a)^{AB}\,, \tag{2.45}$$

where — using the doubling trick — the left- and right-moving currents are joined according to the boundary conditions A and B on the respective side of ψ_a^{AB}. This allows us to treat boundary and boundary condition changing operators in identical manner (the difference being akin to that between the adjoint and bi-fundamental representation of the same Lie or vertex operator algebra). In particular, all relations derived in this paper are also true if one includes boundary changing sectors, provided that one adds labels for the various boundary sectors in the appropriate places. Note that for each boundary component, the boundary labels must be "cyclically closed" in correlation functions, for example correlators such as $\langle \psi_{a_1} \dots \psi_{a_n} \rangle$ should be expanded to $\langle \psi_{a_1}^{A_1A_2} \dots \psi_{a_n}^{A_nA_1} \rangle$ when restoring boundary labels. In the presence of boundary condition changing sectors, the various algebraic structures extracted in this paper are promoted to their category-theoretic counterparts. This follows in standard manner by viewing D-branes (a.k.a boundary sectors) as objects of a category and identifying boundary and boundary condition changing operators with endomorphisms and morphisms between distinct objects.

3. Immediate properties of tree-level amplitudes

In this section, we discuss the most basic properties of open-closed amplitudes on the disk. After explaining the regularization used in later sections, we show that two basic forms for such amplitudes are equal up to sign and independent of the positions of boundary insertions and, more generally, of the worldsheet metric. Moreover, we check that such amplitudes are cyclic with respect to boundary insertions and completely symmetric with respect to insertions of bulk operators. All properties established in this section are elementary, though the precise proofs in conformal field theory are not always obvious. The main point of interest for later sections is the regularization of open-closed tree-level amplitudes, which will play a central role in our discussion of the algebraic constraints.

3.1 The regularized amplitudes

Since disk amplitudes with integrated boundary descendants are affected by contact divergences, the conformal field theory arguments of later sections will require a regulator.

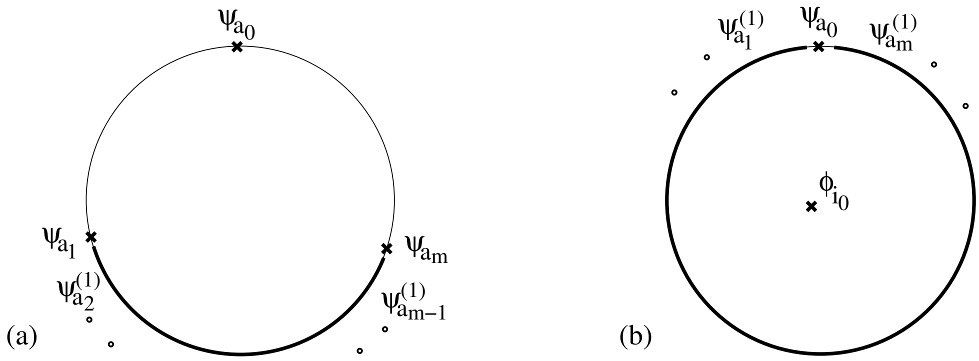

Figure 1: Boundary and bulk insertions for disk amplitudes. (*a*) Three boundary fields ψ_{a_0}, ψ_{a_1} and ψ_{a_m} are at fixed positions, the others are integrated in a path ordered way between ψ_{a_1} and ψ_{a_m}. (*b*) One bulk and one boundary field are fixed. In both cases additional bulk operators may be present, which are integrated over the whole disk.

In this paper, we shall use a version of point-splitting for integrated bulk operators approaching the boundary of the disk and for integrated boundary operators approaching each other. This regularization is essential only for the arguments of sections 5 and 6.1.

Given bulk descendants $\phi^{(2)}_{i_k}$ with $k = 1 \dots n$, we will choose their integration domain as follows:

$$\mathbb{H}_n = \Big\{(z_1, \dots, z_n) \in \mathbb{C}^n \,|\, \Im(z_k) \in (k\epsilon, \infty) \quad \text{for all } k = 1 \dots n\Big\}, \tag{3.1}$$

Here z_k are the insertion points of $\phi^{(2)}_{i_k}$, which of course are integrated over.

We next consider boundary insertions. Using PSL$(2,\mathbb{R})$-invariance, three of them can be fixed while the others are integrated (see figure 1*a*). A typical disk amplitude has the form:

$$\Big\langle \psi_{a_1}\psi_{a_2} P \int \psi^{(1)}_{a_3} \dots \int \psi^{(1)}_{a_{m-1}}\, \psi_{a_m} \int \phi^{(2)}_{i_1} \dots \int \phi^{(2)}_{i_n} \Big\rangle\,, \tag{3.2}$$

where we fixed the positions of ψ_{a_1}, ψ_{a_2} and ψ_{a_m} to the points $\tau_1, \tau_2, \tau_m \in \mathbb{R}$, with the restriction $\tau_1 < \tau_2 < \tau_m$. The path-ordering symbol P means that the integral over $\tau_3 \dots \tau_{m-1}$ runs between τ_2 and τ_m with the constraint $\tau_2 < \tau_3 < \dots < \tau_{m-1} < \tau_m$. Including a regulator, the exact integration domain will be chosen as follows:

$$\mathbb{S}_m(\tau_2, \tau_m) = \Big\{(\tau_3, \dots, \tau_{m-1}) \in \mathbb{R}^{m-3} \,|\, \tau_k - \tau_j > [2(k-j)-1]\epsilon \qquad \text{for } 2 \le j < k \le m\Big\}. \tag{3.3}$$

Remark. Notice that we are requiring slightly increased separations for non-consecutive boundary insertions, rather than working with the naive point-splitting constraint $|\tau_k - \tau_j| \ge |k-j|\epsilon$. This somewhat unusual choice is made for the following reason. The factorization procedure of the following sections makes use of the descent equation $[Q, G] = d/d\tau$, which implies that acting with Q on an integrated boundary insertion produces terms involving the associated zero-form operator evaluated at the boundaries of its integration interval, generally with some integrated insertions squeezed in. The increased separations chosen in (3.3) ensure the presence of non-void integration domains for the squeezed-in operators. For instance, if we consider the BRST operator acting on $\int \psi_{a_4}^{(1)}$, then our choice for the integration domain $\mathbb{S}_m(\tau_2, \tau_m)$ leads to a term of the form:

$$\psi_{a_2}(\tau_2) \int_{\tau_2+\epsilon}^{\tau_4-\epsilon} d\tau_3\ \psi_{a_3}^{(1)}(\tau_3)\ \psi_{a_4}(\tau_4)\Big|_{\tau_4=\tau_2+3\epsilon} = \psi_{a_2}(\tau_2) \int_{\tau_2+\epsilon}^{\tau_2+2\epsilon} d\tau_3\ \psi_{a_3}^{(1)}(\tau_3)\ \psi_{a_4}(\tau_2+3\epsilon)\,,$$

which involves integration over a non-void interval. Had we used the naive condition $|\tau_k - \tau_l| > |k-l|\epsilon$, the integral in the last equation would have been $\int_{\tau_2+\epsilon}^{\tau_2+\epsilon} \psi_{a_3}^{(1)} = 0$.

Besides (3.2) one can also consider amplitudes in which PSL$(2,\mathbb{R})$-invariance is used to fix the positions of one bulk and one boundary insertion (see figure 1*b*):

$$\left\langle \phi_{i_1}\psi_{a_1}\, P \int \psi_{a_2}^{(1)} \ldots \int \psi_{a_m}^{(1)} \int \phi_{i_2}^{(2)} \ldots \int \phi_{i_n}^{(2)} \right\rangle. \tag{3.4}$$

Naively, the integration domain is obtained from $\mathbb{S}_m(\tau_2, \tau_m)$ by replacing both τ_2 and τ_m by τ_1, where we integrate over the real line and identify $-\infty$ and ∞. However, the integrals approach ψ_{a_1} from both sides, so we have to introduce a further cut-off. We will choose the following integration domain:

$$\begin{aligned}\mathbb{S}_m(\tau_1) = \Big\{(\tau_2,\ldots,\tau_m) \in \mathbb{R}^{m-3} | (\tau_2 \ldots \tau_m) &\text{ is cyclically ordered and} \\ &\tau_k - \tau_l > [2(k-l)-1]\epsilon \quad \text{for } \tau_k > \tau_l > \tau_1 \quad \text{or} \quad \tau_1 > \tau_k > \tau_l\,, \\ &\tau_k - \tau_l > [2(k-l+m)-1]\epsilon \quad \text{for } \tau_k > \tau_1 > \tau_l \Big\}.\end{aligned} \tag{3.5}$$

We will see in a moment that (after removing the regulator) the two kinds amplitudes are equal up to sign, as has been argued before in [26].

Having defined the regularized amplitudes, we shall explore the implications of the conformal Ward identities and of the Ward identities for G. Using the doubling trick, one easily proves the relation:

$$\begin{aligned}\oint \xi(z)\langle G(z)\psi_{a_0}\ldots\psi_{a_m}\phi_{i_1}\ldots\phi_{i_n}\rangle &= \sum_{k=0}^{m} \pm\, \xi(\tau_k)\langle \psi_{a_0}\ldots\psi_{a_k}^{(1)}\ldots\psi_{a_m}\phi_{i_1}\ldots\phi_{i_n}\rangle \pm \\ &\pm \sum_{k=0}^{n} \xi(w_k)\langle \psi_{a_0}\ldots\psi_{a_m}\phi_{i_1}\ldots\phi_{i_k}^{(1,0)}\ldots\phi_{i_n}\rangle \pm \\ &\pm \sum_{k=0}^{n} \bar{\xi}(\bar{w}_k)\langle \psi_{a_0}\ldots\psi_{a_m}\phi_{i_1}\ldots\phi_{i_k}^{(0,1)}\ldots\phi_{i_n}\rangle \\ &= 0\,,\end{aligned} \tag{3.6}$$

where $\xi(z) = az^2 + bz + c$ with $a, b, c \in \mathbb{R}$ is a globally-defined holomorphic vector field on the upper half plane and the signs account for the grading on boundary fields. By the doubling trick, the contour integral on the left hand side encircles all fields and their images with respect to the real axis in the complex plane (which is viewed as a double cover of the upper half plane). In the right hand side we evaluated the residue at every insertion, including the images. The terms containing $\phi_i^{(1,0)}$ arise from the residue at ϕ_i, while the terms containing $\phi_i^{(0,1)}$ arise from the residues at the the images of these insertions.

In the bulk sector, a similar identity implies constancy of the bulk topological metric along the moduli space and integrability of the deformed amplitudes. Below, we will study the consequences of (3.6).

3.2 Equivalence of the two types of amplitudes

We start by explaining the relation between the two kinds of disk amplitudes (3.2) and (3.4). We will show that these a priori different quantities are in fact equal up to sign factors. This was already discussed in [26] and we shall review the argument below in order to extract the correct signs for the case of boundary fields with different degrees. The derivation uses the Ward identities of G to relate integration over a bulk descendant with two integrations over boundary descendants.

As an example, consider the amplitudes $\langle\psi_a\psi_b\psi_c\int\phi_i^{(2)}\rangle$ and $\langle\phi_i\psi_a\ P\int\psi_b\int\psi_c\rangle$. We use the Ward identities:

$$\oint\xi_3\langle G\oint\xi_2 G\ \psi_a(\tau_1)\ \psi_b(\tau_2)\ \psi_c(\tau_3)\ \phi_i(z,\bar z)\rangle = 0\,,$$

and:

$$\oint\xi_3\langle G\psi_a\psi_b\psi_c^{(1)}\phi_i\rangle = 0\,,\qquad \oint\xi_2\langle G\psi_a\psi_b^{(1)}\psi_c\phi_i\rangle = 0\,,$$

with the following choice for the global holomorphic vector fields:

$$\xi_2(z) = (z-\tau_1)(z-\tau_3)\qquad\text{and}\qquad \xi_3(z) = (z-\tau_1)(z-\tau_2)\,.$$

We assume the ordering $\tau_1<\tau_2<\tau_3$. Using equation (3.6), we obtain:

$$\frac{\xi_2(z)\bar\xi_3(\bar z)-\bar\xi_2(\bar z)\xi_3(z)}{\xi_2(\tau_2)\xi_3(\tau_3)}\,\langle\psi_a\psi_b\psi_c\phi_i^{(2)}\rangle = (-1)^{\tilde b}\langle\psi_a\psi_b^{(1)}\psi_c^{(1)}\phi_i\rangle\,. \tag{3.10}$$

The conformal Ward identities ensure that both sides of equation (3.10) depend only on the cross-ratio $\zeta=\frac{(z-\tau_3)(\tau_2-\tau_1)}{(z-\tau_2)(\tau_3-\tau_1)}$ and its complex conjugate. Using the relations:

$$\xi_i(\tau_i)\frac{\partial\zeta}{\partial\tau_i}+\xi_i(z)\frac{\partial\zeta}{\partial z} = 0\,,\quad \text{for } i=2,3\,,$$

we find:

$$\frac{\xi_2(z)\bar\xi_3(\bar z)-\bar\xi_2(\bar z)\xi_3(z)}{\xi_2(\tau_2)\xi_3(\tau_3)} = \left(\frac{\partial\zeta}{\partial z}\frac{\partial\bar\zeta}{\partial\bar z}\right)^{-1}\left(\frac{\partial\zeta}{\partial\tau_2}\frac{\partial\bar\zeta}{\partial\tau_3}-\frac{\partial\zeta}{\partial\tau_3}\frac{\partial\bar\zeta}{\partial\tau_2}\right)^{-1}.$$

Hence the prefactor in equation (3.10) is the jacobian of the coordinate transformation from $(z,\bar z)$ to (τ_2,τ_3). Notice that we are free to rescale bulk and boundary descendants via $\phi^{(2)}\to\lambda_{bulk}\phi^{(2)}$ and $\psi^{(1)}\to\lambda_{bound}\psi^{(1)}$. Taking this into account, we find:

$$(-1)^{\tilde b}\left\langle\psi_a\psi_b\psi_c\int\phi_i^{(2)}\right\rangle = -\left\langle\phi_i\,\psi_a\,P\int\psi_b^{(1)}\int\psi_c^{(1)}\right\rangle, \tag{3.11}$$

where we chose the relative normalization factor to be -1. Of course, this locks the rescalings together through the relation $\lambda_{bulk}\propto\lambda_{bound}^2$.

One can easily generalize the analysis to arbitrary numbers of bulk and boundary insertions. This gives:[6]

$$\begin{aligned} B_{a_0\ldots a_m;i_1\ldots i_n} &:= (-1)^{\tilde{a}_1+\ldots+\tilde{a}_{m-1}} \left\langle \psi_{a_0}\psi_{a_1}\, P \int \psi^{(1)}_{a_2} \ldots \int \psi^{(1)}_{a_{m-1}}\, \psi_{a_m} \int \phi^{(2)}_{i_1} \ldots \int \phi^{(2)}_{i_n} \right\rangle \\ &= -\left\langle \phi_{i_1}\psi_{a_0}\, P \int \psi^{(1)}_{a_1} \ldots \int \psi^{(1)}_{a_m} \int \phi^{(2)}_{i_2} \ldots \int \phi^{(2)}_{i_n} \right\rangle . \end{aligned} \tag{3.12}$$

Thus (3.2) and (3.4) are equal up to sign, and they determine the single object $B_{a_0\ldots a_m;i_1\ldots i_n}$ defined by the expression above. These amplitudes vanish unless:

$$\tilde{a}_0 + \cdots + \tilde{a}_m = \tilde{\omega}\,. \tag{3.13}$$

As we shall see below, it is notationally convenient to define:

$$B_{a_0a_1} = B_{a_0} = B_i = 0\,. \tag{3.14}$$

We make one final remark about equation (3.12). The first line is manifestly symmetric in the bulk indices, but this is not obvious for the second line. As in the pure bulk theory, there exists a Ward identity [12, 26] which switches fixed and integrated bulk insertions. This can be used to show directly that the second line in (3.12) is also totally symmetric in the bulk insertions.

For later reference, let us translate the topological sewing constraints of Subsection 2.2.1 in terms of the amplitudes $B_{a_0\ldots a_m;i_1\ldots i_n}$. From equation (3.12), we have:

$$B_{abc} = (-1)^b D_{abc}\,, \qquad B_{a;i} = -e_{ia} = -f_{ai}\,.$$

Introducing the quantities:

$$B^a{}_{a_1\ldots a_m;i_1\ldots i_n} := \omega^{ab} B_{ba_1\ldots a_m;i_1\ldots i_n}\,, \qquad B^i{}_{a_0\ldots a_m;i_2\ldots i_n} := \eta^{ij} B_{a_0\ldots a_m;j,i_2\ldots i_n}\,, \tag{3.16}$$

we find the relations:

$$B_{0ab} = (-1)^{\tilde{a}}\omega_{ab}\,, \qquad B^a_{bc} = (-1)^{\tilde{b}} D^a_{bc}\,, \qquad B^a_i = -e^a_i\,, \qquad B^i_a = -f^i_a\,.$$

Thus equations (2.17), (2.31), (2.32) and (2.34) take the form:

$$B^{a_0}_{ca_3} B^c_{a_1a_2} = -(-1)^{\tilde{a}_1} B^{a_0}{}_{a_1c} B^c{}_{a_2a_3} \tag{3.17}$$

$$B^c_{ab} B^b_i = -B^b_i B^c_{ba} \tag{3.18}$$

$$B^c_{ab} B^a_i B^b_j = C^k_{ij}\, B^c_k \tag{3.19}$$

$$\eta^{ij} B_{ia} B_{jb} = (-1)^{s(c,d)} B^c_{ad} B^d_{cb}\,, \tag{3.20}$$

where the sign factor s depends on c, d.

[6] Notice that for $B_{a_0a_1a_2}$ as well as $B_{a_0;i_1}$ and $B_{a_0a_1;i_1}$ such a relation does not exist for obvious reasons.

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3.3 Two point correlation functions are not deformed

In this subsection, we show that the two-point boundary correlators are constant under bulk and boundary deformations. Let us start with the Ward identity for G in the presence of two fixed boundary insertions:

$$\oint \xi(z)\big\langle G(z)\,\psi_{a_1}(\tau_1)\psi_{a_2}(\tau_2)\psi_{a_3}(\tau_3)\big\rangle = 0\,.$$

Choosing $\xi(z) = (z-\tau_1)(z-\tau_2)$, we find:

$$\big\langle \psi_{a_1}\,\psi_{a_2}\,\psi^{(1)}_{a_3}\big\rangle = 0\,. \tag{3.22}$$

The analogous relation for a bulk perturbation:

$$\big\langle \psi_{a_1}\,\psi_{a_2}\,\phi^{(2)}_{i}\big\rangle = 0\,, \tag{3.23}$$

requires a bit more work. For this, consider the Ward identity:

$$\oint \xi_2\big\langle G \oint \xi_1 G\psi_{a_1}(\tau_1)\psi_{a_2}(\tau_2)\phi_i(w,\bar{w})\big\rangle = 0\,,$$

where $\xi_1(z) = (z-\tau_1)(z-\tau_2)$ and $\xi_2(z) = (z-\tau_2)(z-\Re w)$. Combining this with the relation:

$$\oint \xi_1\big\langle G\psi_{a_1}\psi^{(1)}_{a_2}\phi_i\big\rangle = 0\,,$$

leads to equation (3.23).

Since the supercharge G does not act on additional descendants $\int\psi^{(1)}_a$ and $\int\phi^{(2)}_i$, we easily infer the generalization:

$$\left\langle \psi_{a_1}\,\psi_{a_2}\,P\int\psi^{(1)}_{a_3}\cdots\int\psi^{(1)}_{a_m}\int\phi^{(2)}_{i_1}\cdots\int\phi^{(2)}_{i_n}\right\rangle = 0\,, \qquad \text{for } m\geq 3 \qquad \text{or} \qquad n\geq 1\,. \tag{3.26}$$

In similar manner, one shows:

$$\begin{aligned}\left\langle \phi^{(1,0)}_{i_0}\,\psi_{a_1}\,P\int\psi^{(1)}_{a_2}\cdots\int\psi^{(1)}_{a_m}\int\phi^{(2)}_{i_1}\cdots\int\phi^{(2)}_{i_n}\right\rangle &= 0\,,\\ \left\langle \phi^{(0,1)}_{i_0}\,\psi_{a_1}\,P\int\psi^{(1)}_{a_2}\cdots\int\psi^{(1)}_{a_m}\int\phi^{(2)}_{i_1}\cdots\int\phi^{(2)}_{i_n}\right\rangle &= 0\,,\end{aligned} \tag{3.27}$$

In terms of the quantities defined in equation (3.12), relation (3.26) takes the form:

$$B_{0a_1\ldots a_m;i_1\ldots i_n} = 0 \qquad \text{for } m\geq 3 \qquad \text{or} \qquad n\geq 1\,. \tag{3.28}$$

The identities discussed in this subsection will be important for subsequent arguments. As we shall see, they are relevant for proving independence of the amplitudes of the positions of unintegrated insertions and more generally of the worldsheet metric. Moreover, they relate to special properties of the boundary algebra and topological metric.

3.4 Independence of the positions of unintegrated insertions

We will now show that the fundamental amplitudes (3.12) are independent of the positions of unintegrated insertions. As an example, consider the 4-point boundary amplitude. Differentiating it with respect to τ_1 and using the descent equations, we find:

$$\begin{aligned}\frac{\partial}{\partial\tau_1}\left\langle \psi_{a_0}\psi_{a_1}\int_{\tau_1}^{\tau_3}\psi^{(1)}_{a_2}\psi_{a_3}\right\rangle &= \left\langle \psi_{a_0}[Q,\psi^{(1)}_{a_1}]\int_{\tau_1}^{\tau_3}\psi^{(1)}_{a_2}\psi_{a_3}\right\rangle - \langle\psi_{a_0}\psi_{a_1}\psi^{(1)}_{a_2}|_{\tau_1}\psi_{a_3}\rangle \\ &= (-1)^{\tilde a_1}\langle\psi_{a_0}\psi^{(1)}_{a_1}(\psi_{a_2}|_{\tau_1}-\psi_{a_2}|_{\tau_3})\psi_{a_3}\rangle - \langle\psi_{a_0}\psi_{a_1}\psi^{(1)}_{a_2}|_{\tau_1}\psi_{a_3}\rangle \\ &= (-1)^{\tilde a_1}\Big(\langle\psi_{a_0}(\psi_{a_1}\psi_{a_2})^{(1)}\psi_{a_3}\rangle - \langle\psi_{a_0}\psi^{(1)}_{a_1}(\psi_{a_2}\psi_{a_3})\rangle\Big) = 0\end{aligned} \tag{3.29}$$

In the last line we used relation (3.26). Generalizing this argument, it is not hard to show that all amplitudes (3.12) are independent on the positions of unintegrated insertions.

3.5 Independence of the worldsheet metric

Due to the nontrivial terms in the right hand side of equation (2.40), it is not immediately clear that the amplitudes (3.12) are independent of the worldsheet metric. The usual recipe of topological field theory does not work: the variation of the correlation function with respect to the metric produces an insertion of the energy-momentum tensor, which can be written as $[Q, G_{\mu\nu}]$. When pulling the BRST operator through the integrated boundary insertions, one obtains nontrivial terms induced by equation (2.40), so one cannot immediately conclude that integrated correlators are independent of the worldsheet metric. However, conformal invariance comes to the rescue, through the conformal Ward identity:

$$\begin{aligned}\langle T(z)\ \phi_{i_1}\dots\phi_{i_n}\ \psi_{a_1}\dots\psi_{a_m}\rangle &= \sum_{k=1}^{n}\left(\frac{h_k}{(z-z_k)^2}+\frac{1}{z-z_k}\frac{\partial}{\partial z_k}\right)\langle\phi_{i_1}\dots\phi_{i_n}\psi_{a_1}\dots\psi_{a_m}\rangle + \\ &+\sum_{k=1}^{n}\left(\frac{\bar h_k}{(z-\bar z_k)^2}+\frac{1}{z-\bar z_k}\frac{\partial}{\partial\bar z_k}\right)\langle\phi_{i_1}\dots\phi_{i_n}\psi_{a_1}\dots\psi_{a_m}\rangle + \\ &+\sum_{l=1}^{m}\left(\frac{h_l}{(z-\tau_l)^2}+\frac{1}{z-\tau_l}\frac{\partial}{\partial\tau_l}\right)\langle\phi_{i_1}\dots\phi_{i_n}\psi_{a_1}\dots\psi_{a_m}\rangle\end{aligned} \tag{3.30}$$

where $\phi_{i_k}=\phi_{i_k}(z_k,\bar z_k)$ and $\psi_{a_l}=\psi_{a_l}(\tau_l)$ are bulk and boundary conformal primaries. In the case of interest, the conformal weights of zero-form operators are $h_l=h_k=\bar h_k=0$, while for descendants one has $h_l=h_k=\bar h_k=1$.

Let us first consider the simplest case, namely the boundary 4-point amplitude:

$$\begin{aligned}\int_{\tau_2}^{\tau_4} d\tau_3\ \langle T(z)\ \psi_{a_1}\psi_{a_2}\psi^{(1)}_{a_3}\psi_{a_4}\rangle &= \sum_{l=1,2,4}\int_{\tau_2}^{\tau_4} d\tau_3\frac{1}{z-\tau_l}\frac{\partial}{\partial\tau_l}\langle\psi_{a_1}\psi_{a_2}\psi^{(1)}_{a_3}\psi_{a_4}\rangle + \\ &+\int_{\tau_2}^{\tau_4} d\tau_3\ \frac{\partial}{\partial\tau_3}\left(\frac{1}{z-\tau_3}\langle\psi_{a_1}\psi_{a_2}\psi^{(1)}_{a_3}\psi_{a_4}\rangle\right).\end{aligned}$$

Using the descent relations (2.38) in the first line and evaluating the integral in the second line, we find:

$$\int_{\tau_2}^{\tau_4} d\tau_3\langle T(z)\psi_{a_1}\psi_{a_2}\psi^{(1)}_{a_3}\psi_{a_4}\rangle = (-1)^{\tilde a_1+\tilde a_2}\frac{1}{z-\tau_1}\Big(\langle\psi^{(1)}_{a_1}\psi_{a_2}(\psi_{a_3}\psi_{a_4})\rangle -$$

$$
\begin{aligned}
& -\left\langle\psi_{a_1}^{(1)}(\psi_{a_2}\psi_{a_3})\psi_{a_4}\right\rangle\Big)+(-1)^{\tilde{a}_2}\frac{1}{z-\tau_2}\times \\
& \times\left(\left\langle\psi_{a_1}(\psi_{a_2}\psi_{a_3})^{(1)}\psi_{a_4}\right\rangle-\left\langle\psi_{a_1}\psi_{a_2}^{(1)}(\psi_{a_3}\psi_{a_4})\right\rangle\right)+ \\
& +\frac{1}{z-\tau_3}\left(\left\langle\psi_{a_1}\psi_{a_2}(\psi_{a_3}\psi_{a_4})^{(1)}\right\rangle-\left\langle\psi_{a_1}(\psi_{a_2}\psi_{a_3})\psi_{a_4}^{(1)}\right\rangle\right) \\
= {} & 0\,.
\end{aligned}
$$

In the last step, we used again equation (3.26). In the same manner one can show that all amplitudes (3.12) are independent of the worldsheet metric.

3.6 Cyclicity and bulk permutation invariance

We shall now prove that disk correlation functions are (graded) cyclically symmetric with respect to boundary insertions and symmetric under arbitrary permutations of bulk insertions.

Let us illustrate this with the boundary 4-point amplitude:

$$
\oint \xi(z)\left\langle G(z)\;\psi_{a_1}(\tau_1)\psi_{a_2}(\tau_2)\psi_{a_3}(\tau_3)\psi_{a_4}(\tau_4)\right\rangle=0\,, \tag{3.31}
$$

where $\tau_4 > \cdots > \tau_1$. Taking $\xi(z) = (z-\tau_4)(z-\tau_1)$ in equation (3.31) and using relation (3.6), we obtain $\xi(\tau_2)\langle\psi_a\psi_b^{(1)}\psi_c\psi_d\rangle = (-1)^{\tilde{b}}\xi(\tau_3)\langle\psi_a\psi_b\psi_c^{(1)}\psi_d\rangle$. From the conformal Ward identities we know that the unintegrated 4-point function depends only on the cross-ratio $\zeta = \frac{(\tau_4-\tau_3)(\tau_2-\tau_1)}{(\tau_4-\tau_2)(\tau_3-\tau_1)}$, which satisfies the relation:

$$
\xi(\tau_2)\frac{\partial\zeta}{\partial\tau_2}+\xi(\tau_3)\frac{\partial\zeta}{\partial\tau_3}=0\,.
$$

Hence the Ward identity (3.31) implies:

$$
\left(\frac{\partial\zeta}{\partial\tau_2}\right)^{-1}\langle\psi_a\psi_b^{(1)}\psi_c\psi_d\rangle=-(-1)^{\tilde{b}}\left(\frac{\partial\zeta}{\partial\tau_3}\right)^{-1}\langle\psi_a\psi_b\psi_c^{(1)}\psi_d\rangle\,.
$$

Let us integrate this equation over ζ, taking into account that on the right-hand side the integration runs in the 'wrong' direction, i.e. $\int_0^1 d\zeta(\frac{\partial\zeta}{\partial\tau_2})^{-1} = \int_{\tau_1}^{\tau_3} d\tau_2$, but $\int_0^1 d\zeta(\frac{\partial\zeta}{\partial\tau_3})^{-1} = -\int_{\tau_2}^{\tau_4} d\tau_3$. This gives the relation:

$$
\langle\psi_a P\int\psi_b^{(1)}\psi_c\psi_d\rangle=(-1)^{\tilde{b}}\langle\psi_a\psi_b P\int\psi_c^{(1)}\psi_d\rangle\,.
$$

Generalizing the argument to more integrated insertions, one finds the following identities:

$$
\begin{aligned}
&\left\langle\psi_{a_0}\psi_{a_1}\,P\int\psi_{a_2}^{(1)}\ldots\int\psi_{a_{m-1}}^{(1)}\psi_{a_m}\right\rangle= \\
&\qquad=(-1)^{\tilde{a}_1+\ldots+\tilde{a}_{m-2}}\left\langle\psi_{a_0}\,P\int\psi_{a_1}^{(1)}\ldots\int\psi_{a_{m-2}}^{(1)}\psi_{a_{m-1}}\psi_{a_m}\right\rangle
\end{aligned} \tag{3.33}
$$

and:

$$\left\langle \phi_i \psi_{a_0} P \int \psi^{(1)}_{a_1} \ldots \int \psi^{(1)}_{a_m} \right\rangle = (-1)^{\tilde{a}_0+\ldots+\tilde{a}_{m-1}} \left\langle \phi_i\, P \int \psi^{(1)}_{a_0} \ldots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m} \right\rangle . \quad (3.34)$$

Additional bulk perturbations do not change these results. We conclude that the fundamental disk amplitudes $B_{a_0\ldots a_m;i_1\ldots i_n}$ with $m, n \geq 0$ and $2n + m > 1$ are cyclically symmetric in the boundary indices:

$$B_{a_0\ldots a_m;i_1\ldots i_n} = (-1)^{\tilde{a}_m(\tilde{a}_0+\ldots+\tilde{a}_{m-1})} B_{a_m a_0\ldots a_{m-1};i_1\ldots i_n} . \quad (3.35)$$

Moreover, all such amplitudes are totally symmetric in the bulk indices (the argument is the same as for the pure bulk case [12]).

4. The effective superpotential

In this section, we explain how one can package open-closed disk amplitudes into a generating function, and how this relates to the effective superpotential mentioned in the introduction.

4.1 Deformed amplitudes on the disk

The last statement of the previous subsection implies that we can integrate all bulk perturbations to produce generating functions:

$$\mathcal{F}_{a_0\ldots a_m}(t) \qquad \text{for } m \geq 0 \quad (4.1)$$

with the following property:

$$B_{a_0\ldots a_m;i_1\ldots i_n} = \partial_{i_1} \ldots \partial_{i_n} \mathcal{F}_{a_0\ldots a_m}(t)|_{t=0} . \quad (4.2)$$

For $m \geq 2$, the generating functions are given by the expressions:

$$\mathcal{F}_{a_0\ldots a_m}(t) = (-1)^{\tilde{a}_1+\cdots+\tilde{a}_{m-1}} \left\langle \psi_{a_0}\psi_{a_1} P \int \psi_{a_2} \ldots \int \psi_{a_{m-1}} \psi_{a_m} e^{\sum_p t_p \int_{D^2} \phi^{(2)}_p} \right\rangle ,$$

which are understood as the formal power series:

$$\begin{aligned} \mathcal{F}_{a_0\ldots a_m}(t) &= (-1)^{\tilde{a}_1+\cdots+\tilde{a}_{m-1}} \times \quad (4.4) \\ &\times \sum_{N_0\ldots N_{h_c-1}=0}^{\infty} \prod_{p=0}^{h_c-1} \frac{t_p^{N_p}}{N_p!} \left\langle \psi_{a_0}\psi_{a_1} P \int \psi_{a_2} \ldots \int \psi_{a_{m-1}} \psi_{a_m} \left[\int \phi^{(2)}_p \right]^{N_p} \right\rangle . \end{aligned}$$

The cases $m = 0$ and $m = 1$ of (3.4) are special, because one bulk operator is not integrated. However, through the Ward identity for G, such correlators are again totally symmetric in the bulk indices. Thus one can define $\mathcal{F}_a(t)$ and $\mathcal{F}_{ab}(t)$ through the relations:

$$\partial_i \mathcal{F}_a(t) = -\left\langle \phi_i\, \psi_a\, e^{\sum_p t_p \int_{D^2} \phi^{(2)}_p} \right\rangle ,$$

$$\partial_i \mathcal{F}_{ab}(t) = -\left\langle \phi_i \; \psi_a P \int \psi_b^{(1)} \; e^{\sum_p t_p \int_{D^2} \phi_p^{(2)}} \right\rangle , \tag{4.5}$$

which determine these quantities up to t-independent terms.

Cyclicity of disk amplitudes with respect to boundary insertions (equation (3.35)) implies:

$$\mathcal{F}_{a_0 \ldots a_m}(t) = (-1)^{\tilde{a}_m(\tilde{a}_0 + \ldots + \tilde{a}_{m-1})} \mathcal{F}_{a_m a_0 \ldots a_{m-1}}(t) , \tag{4.6}$$

while equations (3.28) give:

$$\mathcal{F}_{0 a_1 \ldots a_m}(t) = 0 \qquad \text{for } m \neq 2 . \tag{4.7}$$

and:

$$\mathcal{F}_{0 a_1 a_2}(t) = \omega_{a_1 a_2} = \text{ independent of } t . \tag{4.8}$$

Mimicking the closed string case reviewed in section 2, we define deformed amplitudes by:

$$B_{a_0 \ldots a_m ; i_1 \ldots i_n}(t) := \partial_{i_1} \ldots \partial_{i_n} \mathcal{F}_{a_0 \ldots a_m}(t) , \tag{4.9}$$

i.e.:

$$\begin{aligned} B_{a_0 \ldots a_m ; i_1 \ldots i_n}(t) \; &= \; (-1)^{\tilde{a}_1 + \cdots + \tilde{a}_{m-1}} \times \\ &\times \left\langle \psi_{a_0} \psi_{a_1} P \int \psi_{a_2}^{(1)} \ldots \int \psi_{a_{m-1}}^{(1)} \psi_{a_m} P \int \phi_{i_1}^{(2)} \ldots \int \phi_{i_n}^{(2)} e^{\sum_p t_p \int_{D^2} \phi_p^{(2)}} \right\rangle . \end{aligned}$$

Notice that $B_a(t) = \mathcal{F}_a(t)$ and $B_{ab}(t) = \mathcal{F}_{ab}(t)$ need not vanish, though they must be of order at least one in t_i (cf. equations (3.14)). In particular, this means that deformations of the closed string background will generally induce tadpoles:

$$B_a(t) := \langle \psi_a \rangle_t ,$$

where $\langle \ldots \rangle_t$ stands for the expectation value on the disk taken in the deformed theory. Such tadpoles must of course be canceled (for example by performing a shift of the boundary topological vacuum) if the deformed theory is to be conformal (and generally a meaningful string background). This means that deformations of the bulk and boundary sectors must be locked together in order to solve the obstructions, a phenomenon well-known from joint deformation theory. We shall further discuss this phenomenon in Subsection 6.1.5, and exemplify it for concrete physical models in section 7.

4.2 The formal generating function and the effective superpotential

It is possible to package the cyclic amplitudes $\mathcal{F}_{a_0 \ldots a_m}$ defined in (4.1) into a single generating function as follows. Consider the noncommutative and associative superalgebra of formal power series $\hat{\mathbb{A}} = \mathbb{C}[[\hat{s}_a]]$ in the variables $\hat{s}_a$ of degrees $\tilde{a} \in \mathbb{Z}_2$, where a runs from 0 to $h_o - 1$. We define the *formal generating function* $\hat{\mathcal{W}}$ through the expression:

$$\hat{\mathcal{W}} = \sum_{m \geq 1} \frac{1}{m} \hat{s}_{a_m} \ldots \hat{s}_{a_1} \mathcal{F}_{a_1 \ldots a_m}(\hat{t}) , \tag{4.11}$$

where $\mathcal{F}_{a_1\dots a_m}(\hat{t})$ are viewed as formal power series. This quantity is an element of the associative superalgebra $\hat{\mathcal{B}} := \mathbb{C}[[\hat{t}]] \otimes \hat{\mathbb{A}}$, where $\mathbb{C}[[\hat{t}]] := \mathbb{C}[[\hat{t}_0 \dots \hat{t}_{h_c-1}]]$ is the algebra of formal power series in the even and commuting variables $\hat{t}_i$.

Since $\hat{s}_a$ are non-commuting, the quantity $\hat{\mathcal{W}}$ has no obvious physical interpretation, so the reader might wonder what is the use of considering non-commuting parameters in the first place. To understand this, notice that we can evaluate (4.11) on supercommuting variables s_a of degrees $\tilde{a}$ (so that $s_a s_b = (-1)^{\tilde{a}\tilde{b}} s_b s_a$). More precisely, consider a morphism of unital superalgebras $\pi : \hat{\mathcal{B}} \to \mathcal{B}$, where $\mathcal{B}$ is a unital Banach *commutative* superalgebra and let $s_a = \pi(\hat{s}_a)$ and $t_a := \pi(\hat{t}_a)$. Then we define the evaluation of $\hat{\mathcal{W}}$ at (s,t) through:

$$\mathcal{W}(s,t) := \sum_{m\geq 1} \frac{1}{m} s_{a_m} \dots s_{a_1} \mathcal{F}_{a_1\dots a_m}(t) \in \mathcal{B}\,, \tag{4.12}$$

where we assume that the series in the right hand side is absolutely convergent. Here $\mathcal{F}_{a_1\dots a_m}(t)$ is the evaluation of $\mathcal{F}_{a_1\dots a_m}$ at t. Formally, we have $\mathcal{W}(s,t) = \pi(\hat{\mathcal{W}})$.

Since s_a super-commute and have the same $\mathbb{Z}_2$-degree as the boundary descendants $\psi_a^{(1)}$, they can be can be viewed as honest boundary deformation parameters of the world-sheet theory. For appropriate choices of π and $\mathcal{B}$, the quantity $\mathcal{W}$ can be viewed as the space-time effective superpotential of the untwisted $N = 2$ model, when such an interpretation of the world-sheet theory is available.

Because s_a super-commute, it follows that monomials in these variables differing by a permutation are related through:

$$s_{a_{\sigma(m)}} \dots s_{a_{\sigma(1)}} = \eta(\sigma; a_1 \dots a_m) s_{a_m} \dots s_{a_1}\,.$$

Here σ is a permutation on n elements and $\eta(\sigma; a_1 \dots a_m)$ is defined as the sign produced when permuting s_a to relate the left and right hand sides. Using this relation, $\mathcal{W}(s,t)$ reduces to:

$$\mathcal{W}(s,t) = \sum_{m\geq 1} \frac{1}{m!} s_{a_m} \dots s_{a_1} \mathcal{A}_{(a_1\dots a_m)}(t)\,, \tag{4.14}$$

where:

$$\mathcal{A}_{a_1\dots a_m}(t) := (m-1)!\mathcal{F}_{(a_1\dots a_m)}(t) := \frac{1}{m} \sum_{\sigma\in S_m} \eta(\sigma; a_1 \dots a_m) \mathcal{F}_{a_{\sigma(1)}\dots a_{\sigma(m)}}(t)$$

are (super-)symmetrized combinations of the cyclic amplitudes $\mathcal{F}_{a_1\dots a_m}(t)$ and S_m is the group of permutations of m objects. These are the relevant, physically observable quantities, because tree-level scattering amplitudes are summed over permutations of indistinguishable incoming states. By construction, these functions are integrable with respect to the boundary deformation parameters, namely they are given by partial derivatives of $\mathcal{W}$:

$$\mathcal{A}_{a_1\dots a_m} = (\partial_{a_1} \dots \partial_{a_m} \hat{\mathcal{W}})(s,t)\Big|_{s=0}\,, \tag{4.16}$$

where $\partial_a := \vec{\partial}/\partial \hat{s}_a$ are the canonical left derivations of $\hat{\mathbb{A}}$ and the right hand side is evaluated at (s,t).

It is clear that $\mathcal{W}(s,t)$ carries less information than the full set of disk amplitudes. In other words, one cannot package the entire information of the topological string theory in this quantity alone. As explained above, one way to encode tree-level world-sheet data without loosing any information is to consider the formal generating function $\hat{\mathcal{W}}$ in (4.11). In practical applications (for example in section 7 below), we shall choose the evaluation map π such that $t_i = \pi(\hat{t}_i) \in \mathbb{C} \cdot 1_{\mathcal{B}}$ for all i and $s_a = \pi(\hat{s}_a) \in 1 \in \mathbb{C} \cdot 1_{\mathcal{B}}$ for all $\hat{s}_a$ of even degree (here $1_{\mathcal{B}}$ is the unit of $\mathcal{B}$). Then the restriction $W(s,t)|_{s^{odd}=0}$, where $s^{odd} = (s_a)_{\tilde{a}=odd}$ defines a function of the complex variables t_i and $(s_a)_{\tilde{a}=even}$.

5. Homotopy associativity constraints on boundary amplitudes on the disk

In this section, we discuss a countable set of algebraic constraints on tree-level boundary amplitudes on the disk, which can be viewed as the Ward identities of the BRST symmetry. These constraints arise from the relations [26]:

$$\langle [Q, \psi_{a_0}\psi_{a_1} P \int \psi_{a_2}^{(1)} \dots \int \psi_{a_{m-1}}^{(1)} \psi_{a_m}] \rangle = 0 \qquad (m \geq 2)\,, \tag{5.1}$$

which encode BRST invariance of the topological vacuum. They are due to equation (2.40), which induces nontrivial contributions when taking the commutator with the BRST operator in the left hand side of (5.1). From (2.40), it is clear that the resulting terms will involve amplitudes in which two boundary insertions approach each other in the limit when the regulator ϵ is removed. Therefore, the contribution on the left hand side of (5.1) is due entirely to contact singularities, and hence it can be factorized into amplitudes with lower numbers of insertions. Performing the computation, one finds that the Ward identities of the BRST symmetry can be brought to a form known in the mathematics literature as a "minimal A_∞ algebra".

Before proceeding with the computation, we briefly mention another, and perhaps more fundamental, point of view. Since the contributions in the left hand side of (5.1) arise entirely from contact terms, it is clear that their factorization in the limit $\epsilon \to 0$ is intimately connected with an appropriate choice of compactification of the moduli space of disks with boundary markings. As in the closed string case, the appropriate compactification is provided by so-called "stable disks", which describe the allowed degenerations of such geometric objects. In the limit $\epsilon \to 0$, factorization of the terms produced on the right hand side of (5.1) corresponds to a contribution to the disk amplitude coming from the boundary of this compactified moduli space. Writing the amplitude as the integral of a closed differential form over this space, equation (5.1) amounts to the statement that this boundary contribution must vanish. Hence the A_∞ structure can be viewed as a consequence of the topology of this boundary. In abstract terms, it arises because the strata of the stable compactification obey the defining rules of the so-called "little intervals operad" with respect to the composition law induced by sewing of stable disks at their boundary punctures. This point of view on the origin of the A_∞ constraints is intimately connected with open string field theory in its general formulation given by Zwiebach (see [34] and

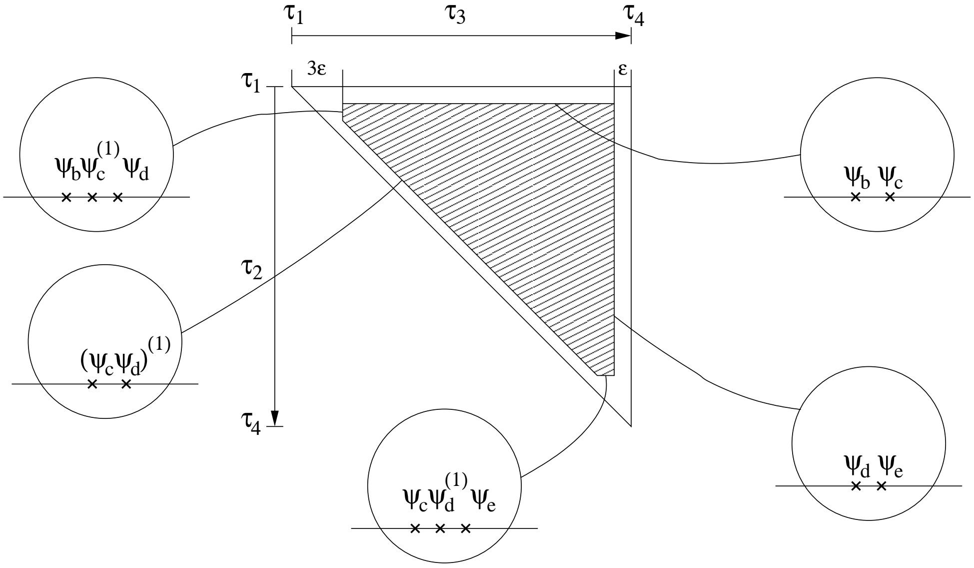

Figure 2: The integration domain $\mathbb{S}_5(\tau_2, \tau_5)$ and its boundary components (through a magnifying glass) for the correlation function $\langle \psi_a(\tau_0)\psi_b(\tau_1)\, P \int \psi_c^{(1)}(\tau_2) \int \psi_d^{(1)}(\tau_3)\, \psi_e(\tau_4)\rangle$.

references therein). In fact, the string field theory perspective provides maybe the most elegant derivation of such constraints, but it lies outside the scope of the present paper, so we shall give the more elementary derivation based on conformal field theory arguments.

As sketched above, acting explicitly with the BRST operator on the left hand side of equation (5.1) and using the descent relation $[Q, \psi^{(1)}_{a_k}] = [Q, [G, \psi_{a_k}]] = \partial_{\tau_k} \psi_{a_k}$ produces an integration over the boundary of the stable compactification of the moduli space of the boundary-punctured disk, where two or more punctures get together very closely. The discussion of the resulting terms involves the regularization (3.3) in an essential manner.

For clarity, we first discuss the case $m = 4$. The regularized configuration space and its boundary components are shown in figure 2. The left-hand side of equation (5.1) becomes:

$$
\begin{aligned}
&\sum_{k=2}^{m-1} (-1)^{s_k} \left\langle \psi_{a_0}\psi_{a_1} P \int \psi^{(1)}_{a_2} \cdots \int \partial_{\tau_k}\psi_{a_k} \cdots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m} \right\rangle = \\
&\quad = \sum_{k=2}^{m-1} (-1)^{s_k} \int_{\tau_1}^{\tau_m} d\tau_k \Bigg(\partial_{\tau_k} \Bigg\langle \psi_{a_0}\psi_{a_1} \int_{\tau_1}^{\tau_k} \psi^{(1)}_{a_2} \int_{\tau_2}^{\tau_k} \psi^{(1)}_{a_3} \cdots \int_{\tau_{k-2}}^{\tau_k} \psi^{(1)}_{a_{k-1}} \psi_{a_k} \int_{\tau_k}^{\tau_{k+2}} \psi^{(1)}_{a_{k+1}} \times \cdots \\
&\qquad\qquad\qquad\qquad \times \int_{\tau_k}^{\tau_m} \psi^{(1)}_{a_{m-1}} \psi_{a_m} \Bigg\rangle - \\
&\qquad\qquad - \sum_{l=2}^{k-1} \Bigg\langle \psi_{a_0}\psi_{a_1} \int_{\tau_1}^{\tau_k} \psi^{(1)}_{a_2} \cdots \left[\psi^{(1)}_{a_l}\Big|_{\tau_l \to \tau_k} \int_{\tau_l}^{\tau_k} \psi^{(1)}_{a_{l+1}} \cdots \psi_{a_k} \right] \times \\
&\qquad\qquad\qquad\qquad \times \int_{\tau_k}^{\tau_{k+2}} \psi^{(1)}_{a_{k+1}} \cdots \int_{\tau_k}^{\tau_m} \psi^{(1)}_{a_{m-1}} \psi_{a_m} \Bigg\rangle + \\
&\qquad\qquad + \sum_{l=k+1}^{m-1} \Bigg\langle \psi_{a_0}\psi_{a_1} \int_{\tau_1}^{\tau_k} \psi^{(1)}_{a_2} \cdots \left[\psi_{a_k} \cdots \int_{\tau_k}^{\tau_l} \psi^{(1)}_{a_{l-1}} \psi^{(1)}_{a_l}\Big|_{\tau_k \leftarrow \tau_l} \right] \times \cdots
\end{aligned}
$$

$$\times \int_{\tau_k}^{\tau_m} \psi^{(1)}_{a_{m-1}} \psi_{a_m} \Big\rangle\Big)$$

$$= \sum_{k=2}^{m-1} (-1)^{s_k} \Bigg(\Big\langle \psi_{a_0} \psi_{a_1} P \int \psi^{(1)}_{a_1} \cdots \int \psi^{(1)}_{a_{k-1}} \Big[\psi_{a_k}\big|_{\tau_k \to \tau_m} P \int \psi^{(1)}_{a_{k+1}} \cdots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m} \Big] \Big\rangle -$$

$$- \Big\langle \psi_{a_0} \Big[\psi_{a_1} P \int \psi^{(1)}_{a_2} \cdots \int \psi^{(1)}_{a_{k-1}} \psi_{a_k}\big|_{\tau_1 \leftarrow \tau_k} \Big] P \int \psi^{(1)}_{a_{k+1}} \cdots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m} \Big\rangle -$$

$$- \sum_{l=2}^{k-1} \Big\langle \psi_{a_0} \psi_{a_1} P \int \psi^{(1)}_{a_2} \cdots \int \Big[\psi_{a_l}\big|_{\tau_l \to \tau_k} P \int \psi^{(1)}_{a_{l+1}} \cdots \psi_{a_k} \Big]^{(1)} \times \cdots$$

$$\times \int \psi^{(1)}_{a_{m-1}} \psi_{a_m} \Big\rangle \Bigg), \tag{5.2}$$

where the sign is given by $s_k = \tilde{a}_0 + \ldots + \tilde{a}_{k-1}$. In the second step we used the fact that the regularized configuration space is a simplex, which means that we have nested integration domains.[7] For notational simplicity, we do not indicate the cut-off ϵ in the integrals.

In the last form of (5.2), the terms in square brackets are products of boundary operators. In the limit $\epsilon \to 0$, we can factorize the result by pulling these terms out while inserting the sum $\sum_{a,b} \psi_c \omega^{cd} \psi_d$ over a basis of the on-shell space of boundary observables. This gives:

$$\sum_{k=2}^{m-1} (-1)^{s_k} \Bigg(\Big\langle \psi_{a_0} \psi_{a_1} P \int \psi^{(1)}_{a_1} \ldots \int \psi^{(1)}_{a_{k-1}} \psi_c \Big\rangle \times$$

$$\times \omega^{cd} \Big\langle \psi_d \psi_{a_k} P \int \psi^{(1)}_{a_{k+1}} \ldots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m} \Big\rangle -$$

$$- \Big\langle \psi_{a_0} \psi_c P \int \psi^{(1)}_{a_{k+1}} \ldots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m} \Big\rangle \omega^{cd} \Big\langle \psi_d \psi_{a_1} P \int \psi^{(1)}_{a_2} \ldots \int \psi^{(1)}_{a_{k-1}} \psi_{a_k} \Big\rangle -$$

$$- \sum_{l=2}^{k-1} \Big\langle \psi_{a_0} \psi_{a_1} P \int \psi^{(1)}_{a_2} \cdots \int \psi^{(1)}_{c} \cdots \int \psi^{(1)}_{a_{m-1}} \psi_{a_m} \Big\rangle \times$$

$$\times \omega^{cd} \Big\langle \psi_d \psi_{a_l} P \int \psi^{(1)}_{a_{l+1}} \ldots \psi_{a_k} \Big\rangle \Bigg) = 0\,. \tag{5.3}$$

We next re-write this equation in terms of the quantities defined in equation (3.16). Using (3.12), we find:

$$\sum_{\substack{k,l=2 \\ k-m+2<l\leq k}}^{m} (-1)^{\tilde{a}_1+\ldots+\tilde{a}_{l-2}} B^b{}_{a_1\ldots a_{l-2} c a_{k+1} \ldots a_m} B^c{}_{a_{l-1}\ldots a_k} = 0 \qquad \text{for } m \geq 2\,. \tag{5.4}$$

In deriving (5.4) we used the selection rules $\tilde{b} = \tilde{a}_1 + \ldots + \tilde{a}_m + 1$ for $B^b{}_{a_1\ldots a_m}$ and $\tilde{\omega} = \tilde{a}_0 + \ldots + \tilde{a}_m$ for $B_{a_0\ldots a_m}$. The restrictions in the sum account for the fact that the

[7]For sake of easier reading, the nested integrals over τ_{k+1} to τ_{m-1} are partly written in the 'wrong' order.

amplitudes $B_{a_0 \dots a_m}$ are considered only for $m \geq 2$ (alternatively, one can remove these constraints and use definitions (3.14)). The first equation in (5.4) is obtained for $m = 2$, and coincides with the associativity condition (3.17) for the boundary product.

5.1 Algebraic description

To make contact with expressions found in the mathematics literature, let us bring (5.4) to a more familiar form. For this, we define tree-level *boundary scattering products* $r_m : H_o^{\otimes m} \to H_o$ to be the multilinear maps determined by the equations:

$$r_m(\psi_{a_1} \dots \psi_{a_m}) = B^{a_0}_{a_1 \dots a_m} \psi_{a_0} \,, \tag{5.5}$$

where, as usual, we use implicit summation over repeated indices. The selection rule for $B^{a_0}_{a_1 \dots a_m}$ gives:

$$\deg r_m(\psi_{a_1} \dots \psi_{a_m}) = 1 + \sum_{j=1}^{m} \tilde{a}_j \,,$$

so all maps r_m have degree one when H_o is endowed with the suspended grading. Equation (5.4) takes the form:

$$\sum_{\substack{k+l=m+1 \\ j=0 \dots k-1}}^{m} (-1)^{\tilde{a}_1 + \dots + \tilde{a}_j} r_k(\psi_{a_1} \dots \psi_{a_j}, r_l(\psi_{a_{j+1}} \dots \psi_{a_{j+l}}), \psi_{a_{j+l+1}} \dots \psi_{a_m}) = 0 \,, \tag{5.7}$$

where we set $r_0 = r_1 = 0$. Relations (5.7) define an A_∞ algebra [67, 68], in conventions in which all products have degree one. For reader's convenience, we summarize the standard terminology concerning such algebras:

(1) A collection of multilinear maps $r_m : H_o^{\otimes m} \to H_o$ of degree $+1$ satisfying (5.7) is called a *weak* A_∞ algebra if m is allowed to run from 0 to ∞.

(2) Such a collection is called a *strong* A_∞ algebra (or simply an A_∞ algebra) if m runs from 1 to infinity.

(3) Such a collection is a *minimal* A_∞ algebra if m runs from 2 to infinity.

Thus a (strong) A_∞ algebra is a weak A_∞ algebra for which $r_0 = 0$, while a minimal A_∞ algebra is a (strong) A_∞ algebra for which $r_1 = 0$. The algebra obtained above is a minimal A_∞ algebra. As we shall see below, bulk perturbations will generically deform this to a *weak* A_∞ algebra. This corresponds to the appearance of a tadpole induced by deformations of the closed string background.

Due to the cyclicity property (3.35) of disk amplitudes, our minimal A_∞ algebra is in fact *cyclic* with respect to the bilinear form on H_o defined by the boundary topological metric. Writing:

$$B_{a_0 \dots a_m} = \omega(\psi_{a_0}, r_m(\psi_{a_1} \dots \psi_{a_m})) \,,$$

this is simply condition (3.35) expressed in terms of string scattering products:

$$\omega(\psi_{a_0}, r_m(\psi_{a_1} \dots \psi_{a_m})) = (-1)^{\tilde{a}_m(\tilde{a}_0 + \dots + \tilde{a}_{m-1})} \omega(\psi_{a_m}, r_m(\psi_{a_0} \dots \psi_{a_{m-1}})) \,. \tag{5.9}$$

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A further constraint follows from equations (3.28), which imply:

$$B^c_{a_1\dots a_{i-1}0a_{i+1}\dots a_m} = 0 \qquad \text{for } m \geq 2 \qquad \text{and all} \quad i = 1\dots m\,,$$

i.e.:

$$r_m(\psi_{a_1}\dots\psi_{a_{i-1}}, 1_o, \psi_{a_{i+1}}\dots\psi_{a_{m-1}}) = 0 \text{for } m \geq 3 \qquad \text{and all} \quad i = 1\dots m-1\,. \tag{5.11}$$

On the other hand, we have:

$$r_2(\psi_a, \psi_b) = B^c_{ab}\psi_c = (-1)^{\tilde{a}} D^c_{ab}\psi_c\,.$$

Using the fact that 1_o is a unit for the boundary algebra, this gives:

$$r_2(1_o, \psi_a) = (-1)^{\tilde{a}} r_2(\psi_a, 1_o) = \psi_a\,. \tag{5.13}$$

Equations (5.11) and (5.13) mean that (H_o, r_*) is a *unital* A_∞ algebra (see, for example, [48]).

Observation. When considering boundary condition changing sectors, the A_∞ algebra discussed above generalizes to an A_∞ category [43].

The relevance of A_∞ algebras was originally pointed out in [29] in the context of open string field theory in the general, non-polynomial formulation given by Zwiebach (see [34] and references therein). In this approach, one obtains A_∞ constraints on open string products. Such products are associated with geometric vertices whose construction depends on a positive parameter l, which measures the length of their external strips. The scattering products considered above can be viewed as the limit $l \to +\infty$ of the string products of [29], while the limit $l \to 0^+$ recovers the better known formulation of [69], in which only the cubic vertex survives. A_∞ algebras were originally introduced by J. Stasheff [67, 68], while A_∞ categories were first discussed by K. Fukaya [43]. They play a central role in the homological mirror symmetry program [42, 43, 45, 46, 44, 47, 48], where they arise via topological string field theory (see [50] and references therein).

6. Constraints on deformed amplitudes

We are now ready to discuss the consistency constraints for mixed bulk-boundary amplitudes on the disk, and derive the generalization of the homotopy associativity constraints of section 5. As we shall see below, the relevant consistency conditions take the form of a weak, cyclic and unital A_∞ algebra, which can be viewed as an all-order deformation of the minimal A_∞ algebra of section 5. The appearance of a weak A_∞ algebra under deformations of the closed string background is due to the generation of an open string tadpole, which must be canceled by a shift of the open string vacuum. This encodes interlocking of open and closed string deformation parameters when solving the joint deformation problem for the bulk and boundary sectors. After discussing the algebraic and physical interpretation of this phenomenon, we investigate the remaining constraints, which encode the stringy generalization of the second bulk-boundary sewing condition and of the Cardy relation. This completes the set of consistency conditions constraining open-closed amplitudes on the disk.

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6.1 Weak A_∞ constraints for mixed amplitudes on the disk

In the present subsection, we extend the discussion of A_∞ constraints to general open-closed amplitudes on the disk. We shall show that the A_∞ structure exhibited in section 5 is promoted to a so-called *weak* A_∞ algebra, which is again cyclic and unital. For simplicity we start by discussing the case of a single boundary insertion. As we shall see below, these amplitudes can be used to define a first order deformation of the A_∞ algebra of section 5, a deformation which preserves cyclicity and unitality but need not preserve minimality. We shall also discuss the general case of multiple insertions, which defines an all-order (formal) deformation in the bulk parameters t_i.

6.1.1 Disk amplitudes with a single bulk insertion

Insertions of bulk operators perturb the minimal A_∞ algebra extracted in section 5. We first consider linear perturbations, which amount to inserting just one bulk operator in the disk amplitudes:

$$\left\langle \left[Q, \phi_i \psi_{a_0} P \int \psi_{a_1}^{(1)} \dots \int \psi_{a_m}^{(1)} \right] \right\rangle = 0 \, . \tag{6.1}$$

As in section 5, acting with the BRST commutator on the integrated descendants on the left hand side produces terms in which several boundary fields approach each other. In the limit $\epsilon \to 0$, we can factorize the result by inserting complete systems of open string observables. Notice that the integration domain for equation (6.1) differs from that of equation (5.1), because we have only one fixed boundary operator. This makes the computation more involved.

Let us illustrate this with the simplest non-trivial case, namely $m = 2$. The integration domain and its boundary components are shown in figure 3. Using the descent equation (2.38), the left-hand side of (6.1) becomes:

$$\begin{aligned}
&(-1)^{a_0} \left\langle \phi_i \psi_{a_0} \int_{\tau_0^+ + \epsilon}^{\tau_0^- - 3\epsilon} \partial_{\tau_1} \psi_{a_1} \int_{L_2(\tau_0,\tau_1)}^{R_2(\tau_0,\tau_1)} \psi_{a_2}^{(1)} \right\rangle + \\
&+ (-1)^{a_0 + a_1 + 1} \left\langle \phi_i \psi_{a_0} \int_{L_1(\tau_0,\tau_2)}^{R_1(\tau_0,\tau_2)} \psi_{a_1}^{(1)} \int_{\tau_0^+ + 3\epsilon}^{\tau_0^- - \epsilon} \partial_{\tau_2} \psi_{a_2} \right\rangle .
\end{aligned} \tag{6.2}$$

The boundary of the integration domain can be inferred from (3.5) and is shown in figure 3. Its components are given by:

$$R_2(\tau_0, \tau_1) = \begin{cases} \tau_0^- - \epsilon & \text{for } \tau_1 > \tau_0^+ + 2\epsilon \\ \tau_0^- - 3\epsilon + (\tau_1 - \tau_0^+) & \text{for } \tau_1 < \tau_0^+ + 2\epsilon \end{cases}$$

$$L_2(\tau_0, \tau_1) = \begin{cases} \tau_1 + \epsilon & \text{for } \tau_1 > \tau_0^+ + 2\epsilon \\ \tau_0^+ + 3\epsilon & \text{for } \tau_1 < \tau_0^+ + 2\epsilon \end{cases} ,$$

with similar expressions for R_1 and L_1. As in the derivation of section 5, we use partial integration taking into account all boundary contributions. Compared to section 5, we have an additional contribution from the upper right corner of the regularized configuration

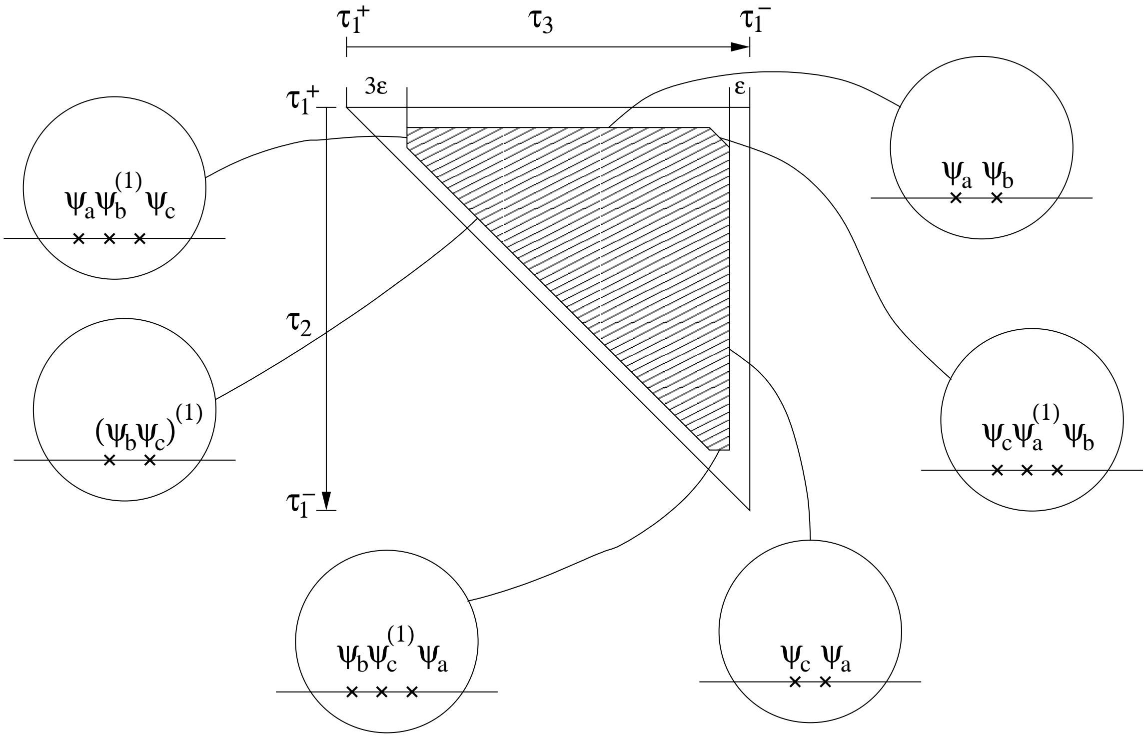

Figure 3: The integration domain $\mathbb{S}_3(\tau_1)$ and its boundary components (through a magnifying glass) for the correlation function $\langle \phi_i(w,\bar{w})\, \psi_a(\tau_1)\, P \int \psi_b(\tau_2) \int \psi_c(\tau_3) \rangle$. The real line, as boundary of the disk, was compactified to a circle by identifying τ_1^+ and τ_1^-.

space in figure 3, which comes from the boundary components R_2 for $\tau_0^+ + \epsilon < \tau_1 < \tau_0^+ + 2\epsilon$ and L_1 for $\tau_0^- - \epsilon > \tau_2 > \tau_0^- - 2\epsilon$. This contribution takes the form:

$$(-1)^{a_0+1+(a_2+1)(a_0+a_1)} \Big\langle \phi_i \int_{\tau_0^+ + \epsilon}^{\tau_0^+ + 2\epsilon} d\tau_1 \Big(\psi_{a_2}^{(1)}(\tau_1 - 3\epsilon) \psi_{a_0}(\tau_0) \psi_{a_1}(\tau_1) + \\ + (-1)^{a_0+a_2} \psi_{a_2}(\tau_1 - 3\epsilon) \psi_{a_0}(\tau_0) \psi_{a_1}^{(1)}(\tau_1) \Big) \Big\rangle = \\ = -(-1)^{a_0+(a_2+1)(a_0+a_1+1)} \Big\langle \phi_i \Big(\psi_{a_2}(\tau_1 - 3\epsilon) \int_{\tau_1 - 2\epsilon}^{\tau_1 - \epsilon} d\tau_0 \psi_{a_0}^{(1)}(\tau_0) \psi_{a_1}(\tau_1) \Big) \Big\rangle ,$$

where we used a Ward identity corresponding to the current G to 'move' the integral from τ_1 to τ_0.[8] Collecting all terms and factorizing as in section 5, we find that relation (6.1)

[8]More precisely, we used the relation:

$$\oint \xi(w) \langle G(w)\ \phi_i(z,\bar{z})\ \psi_{a_2}(\tau_2) \psi_{a_0}(\tau_0)\ \psi_{a_1}(\tau_1) \rangle = 0\,,$$

with $\xi(w) = (w-z)(w-\bar{z})$ as well as the fact that correlators depend only on the cross ratio $\zeta_l = \frac{(z-\bar{z})(\tau_l - \tau_0)}{(z-\tau_l)(\bar{z}-\tau_0)}$ for $l = 1, 2$. In the limit $\tau_2 \to \tau_1 - 3\epsilon$, we have $\zeta_2 = \zeta_1 + \mathcal{O}(\epsilon)$ and we obtain the jacobian $|\frac{\partial \tau_1}{/} \partial \tau_0|$ up to terms $\mathcal{O}(\epsilon)$, which vanish in our limit.

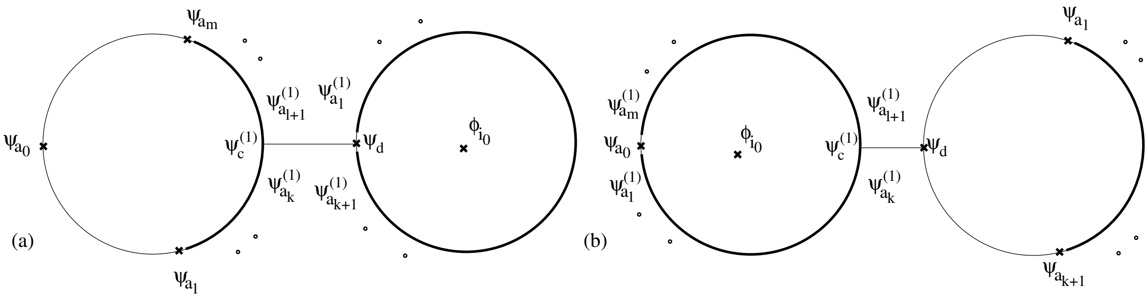

Figure 4: The two contributions to the factorizations leading to equation (6.5). Figure (a) shows a summand of the first term in this equation, while figure (b) shows a summand of the second term.

reduces to:

$$
\begin{aligned}
&(-1)^{\tilde{a}_0}\left\langle \phi_i\psi_c \int \psi^{(1)}_{a_2}\right\rangle \omega^{cd}\langle\psi_d\psi_{a_0}\psi_{a_1}\rangle +\\
&+(-1)^{\tilde{a}_0(\tilde{a}_1+\tilde{a}_2)+\tilde{a}_1+\tilde{a}_2}\langle\phi_i\psi_c\rangle\omega^{cd}\left\langle \psi_d\psi_{a_1}\int\psi^{(1)}_{a_2}\psi_{a_0}\right\rangle +\\
&+(-1)^{\tilde{a}_2(\tilde{a}_0+\tilde{a}_1)+\tilde{a}_2}\left\langle \phi_i\psi_c \int \psi^{(1)}_{a_1}\right\rangle \omega^{cd}\langle\psi_d\psi_{a_2}\psi_{a_0}\rangle +\\
&+(-1)^{\tilde{a}_0+\tilde{a}_1}\left\langle \phi_i\psi_c\right\rangle \omega^{cd}\langle\psi_d\psi_{a_0}\int\psi^{(1)}_{a_1}\psi_{a_2}\rangle +\\
&+(-1)^{\tilde{a}_0+\tilde{a}_2+\tilde{a}_2(\tilde{a}_0+\tilde{a}_1)}\langle\phi_i\psi_c\rangle\omega^{cd}\left\langle \psi_d\psi_{a_2}\int\psi^{(1)}_{a_0}\psi_{a_1}\right\rangle +\\
&+(-1)^{\tilde{a}_0+\tilde{a}_1}\left\langle \phi_i\psi_{a_0} \int \psi^{(1)}_{c}\right\rangle \omega^{cd}\langle\psi_d\psi_{a_1}\psi_{a_2}\rangle = 0\,.
\end{aligned}
\tag{6.3}
$$

When expressed in terms of the quantities defined in (3.12), this equation takes the form:

$$
\begin{aligned}
&B^{a_0}{}_{d;i}B^d{}_{a_1a_2} + B^{a_0}{}_{da_2}B^d{}_{a_1;i} + (-1)^{\tilde{a}_1}B^{a_0}{}_{a_1d}B^d{}_{a_2;i} +\\
&+B^{a_0}{}_{da_1a_2}B^d{}_i + (-1)^{\tilde{a}_1}B^{a_0}{}_{a_1da_2}B^d{}_i + (-1)^{\tilde{a}_1+\tilde{a}_2}B^{a_0}{}_{a_1a_2d}B^d{}_i = 0\,.
\end{aligned}
\tag{6.4}
$$

The general case is a straightforward generalization, but the computations are much more tedious. Therefore, we shall give the result without presenting the details of its proof. It is the natural generalization of (6.4) for $m \geq 1$:

$$
\begin{aligned}
&\sum_{j=0}^{m}\sum_{k=0}^{j}(-1)^{\tilde{a}_1+\ldots+\tilde{a}_k}\,B^{a_0}{}_{a_1\ldots a_k c a_{j+1}\ldots a_m}\,B^c{}_{a_{k+1}\ldots a_j;i}+\\
&+\sum_{j=2}^{m}\sum_{k=0}^{j}(-1)^{\tilde{a}_1+\ldots+\tilde{a}_k}\,B^{a_0}{}_{a_1\ldots a_k c a_{j+1}\ldots a_m;i}\,B^c{}_{a_{k+1}\ldots a_j} = 0\,,
\end{aligned}
\tag{6.5}
$$

where we explicitly set $B^c{}_a = B^c = 0$, consistent with definitions (3.14). The first of equations (6.5) is obtained for $m = 1$ and coincides with the first bulk-boundary sewing constraint (3.18) of TFT.

6.1.2 General disk amplitudes

We now turn to the general case, allowing for an arbitrary number of bulk insertions. Extending the argument of the previous subsection, we will extract a series of constraints which amount to the statement that the deformed disk amplitudes $B_{a_0\ldots a_m}(t) = \mathcal{F}_{a_0\ldots a_m}(t)$ satisfy the defining relations of a weak A_∞ algebra.

Consider a general disk amplitude written in the form:

$$\left\langle [Q, \phi_{i_0}\psi_{a_0} P \int \psi^{(1)}_{a_1} \cdots \int \psi^{(1)}_{a_m} \int \phi^{(2)}_{i_1} \cdots \int \phi^{(2)}_{i_n}] \right\rangle = 0\,. \tag{6.6}$$

Acting with the Q-commutator on the integrated boundary insertions produces a sum over the terms appearing in equation (6.5). Additionally, we also have all contributions from integrated bulk descendants:

$$\sum_{I\subseteq I_{0,n}} \sum_{k\leq j}^{m} (-1)^{\tilde{a}_1+\ldots+\tilde{a}_k} B^{a_0}{}_{a_1\ldots a_k c a_{j+1}\ldots a_m; I_{0,n}\setminus I}\, B^c{}_{a_{k+1}\ldots a_j; I}\,, \tag{6.7}$$

where $I_{p,q} = \{i_p, i_{p+1}, \ldots, i_{q-1}, i_q\}$. Note that the BRST variation of boundary fields does not produce terms containing $B^a{}_{I\setminus i_0}$ and $B^a{}_{b;I\setminus i_0}$. Instead, these missing terms arise from the Q-variation of the integrated bulk insertion:

$$\left[Q, \int \phi^{(2)}_{i_k}\right] = \lim_{\epsilon\to 0} \oint_{k\epsilon} \phi^{(1)}_{i_k}\,. \tag{6.8}$$

The integral runs along a loop, which follows the boundary (namely the real axis) at a distance $k\epsilon$. Using equation (6.8), we obtain contributions from a bulk operator approaching the boundary far away from any boundary operator, and from a bulk operator approaching a boundary insertion. Due to our regularization (3.1), the loop (6.8) cuts the integration domains of the operators $\phi^{(2)}_{i_1} \ldots \phi^{(2)}_{i_{k-1}}$ into a part near the boundary and a bulk part. On the other hand, the operators with $\phi^{(2)}_{i_{k+1}} \ldots \phi^{(2)}_{i_n}$ are inside the loop and hence they don't produce more contact terms. In the limit $\epsilon \to 0$, factorization proceeds by distributing the former operators in all possible ways on the two emerging disks.

As an example, consider the piece of the boundary sitting between ψ_{a_l} and $\psi_{a_{l+1}}$. A typical term produced by the process above has the form:

$$\left\langle \cdots \int \psi_{a_l} \int_{\tau_l}^{\tau_{l+1}} \phi^{(1)}_{i_k} \int \psi_{a_{l+1}} \cdots \prod_{j\neq k} \int \phi^{(2)}_{i_j} \right\rangle.$$

Its factorization produces the contributions $\pm \sum_{I\subseteq I_{1,k-1}} B_{\ldots a_l c a_{l+1}\ldots; i_0 I_{1,n}\setminus\{i_k I\}} B^c{}_{i_k I}$, where we used the notation $\{i_k I\} = \{i_k\}\cup I$. Summing over k leads to a total contribution:

$$\pm \sum_k \sum_{I\subseteq I_{1,k-1}} B_{\ldots a_l c a_{l+1}\ldots; i_0 I_{1,n}\setminus\{i_k I\}} B^c{}_{i_k I} = \pm \sum_{I\subseteq I_{1,n}} B_{\ldots a_l c a_{l+1}\ldots; i_0 I_{1,n}\setminus I} B^c{}_I\,.$$

Similarly, the factorization of a bulk operator approaching an *integrated* boundary field gives rise to the terms:

$$\pm \sum_{I\subseteq I_{1,n}} B_{\ldots a_{l-1} c a_{l+1}\ldots; i_0 I_{1,n}\setminus I} B^c{}_{a_l; I}\,.$$

Finally, bulk operators approaching the fixed insertion ψ_{a_0} produce:

$$\pm \sum_{I\subseteq I_{1,n}} B^{a_0}{}_{c;I_{1,n}\setminus I} B^c{}_{a_1\ldots a_n;i_0 I}\,.$$

This completes the list of contributions from the boundary of the configuration space.

Gathering all terms, we find that equation (6.6) can be written in the following form:

$$\sum_{I\subseteq I_{0,n}} \sum_{\substack{k,j=0\\k\leq j}}^{m} (-1)^{\tilde{a}_1+\ldots+\tilde{a}_k} B^{a_0}{}_{a_1\ldots a_k c a_{j+1}\ldots a_m;I_{0,n}\setminus I}\; B^c{}_{a_{k+1}\ldots a_j;I} = 0\,. \tag{6.13}$$

For $n = 0$, this reduces to equation (6.5) extracted in the previous subsection.

Notice that indices distribute in the same manner as would derivatives with respect to the formal parameters t^j. This means that we can concisely write relations (6.13) as weak A_∞ constraints for the perturbed boundary amplitude $B_{a_0\ldots a_m}(t) := \mathcal{F}_{a_0\ldots a_m}(t)$:

$$\sum_{\substack{k,j=0\\k\leq j}}^{m} (-1)^{\tilde{a}_1+\ldots+\tilde{a}_k} \mathcal{F}^{a_0}{}_{a_1\ldots a_k c a_{j+1}\ldots a_m}(t)\, \mathcal{F}^c{}_{a_{k+1}\ldots a_j}(t) = 0\,. \tag{6.14}$$

Expanding this as a power series in t reproduces equations (6.13). The first two deformed amplitudes: $\mathcal{F}_a$ and $\mathcal{F}_{ab}$ are of order at least one in t_i, since B_a and B_{ab} vanish. The presence of these terms for $t \neq 0$ promotes the minimal A_∞ algebra of section 5 to a weak A_∞ algebra, and corresponds to the generation of non-vanishing tadpoles, as discussed in Subsection 4.1.

6.1.3 Algebraic formulation

Extending the discussion of section 5, let us define *deformed open string scattering products* $r^t_m : H_o^{\otimes m} \to H_o$ through the relations:

$$r^t_m(\psi_{a_1}\ldots\psi_{a_m}) = \mathcal{F}^a_{a_1\ldots a_m}(t)\psi_a \qquad \text{for } m\geq 1\,. \tag{6.15}$$

and:

$$r^t_o(1) := \mathcal{F}^a(t)\psi_a\,,$$

where $\mathcal{F}^a(t) = \omega^{ab}\mathcal{F}_b(t)$ and we used the fact that the product $r_o : H_o^{\otimes 0} \approx \mathbb{C} \to H_o$ is determined by its value at the complex unit $1 \in \mathbb{C}$. As in section 5, equations (6.14) become:

$$\sum_{\substack{k+l=m+1\\j=0\ldots k-1}}^{m} (-1)^{\tilde{a}_1+\cdots+\tilde{a}_j} r^t_k(\psi_{a_1}\ldots\psi_{a_j}, r^t_l(\psi_{a_{j+1}}\ldots\psi_{a_{j+l}}), \psi_{a_{j+l+1}}\ldots\psi_{a_m}) = 0\,, \tag{6.17}$$

which are the standard relations defining a weak A_∞ algebra.

Remembering equation (4.6), we find that this weak A_∞ algebra is cyclic:

$$\omega(\psi_{a_0}, r^t_m(\psi_{a_1}\ldots\psi_{a_m})) = (-1)^{\tilde{a}_m(\tilde{a}_0+\cdots+\tilde{a}_{m-1})}\omega(\psi_{a_m}, r^t_m(\psi_{a_0}\ldots\psi_{a_{m-1}})) \qquad \text{for } m\geq 1\,. \tag{6.18}$$

Moreover, equations (4.7) and (4.8) show that (H_o, r^t_*) is unital:

$$r^t_m(\psi_{a_1}\dots\psi_{a_{i-1}}, 1_o, \psi_{a_{i+1}}\dots\psi_{a_{m-1}}) = 0 \quad \text{for } (m=1 \text{ or } m\geq 3) \quad \text{and all} \quad i = 1\dots m-1 \tag{6.19}$$

and:

$$r^t_2(1_o, \psi_a) = (-1)^{\tilde{a}} r^t_2(\psi_a, 1_o) = \psi_a\,. \tag{6.20}$$

To arrive at the last equation, we used relation (4.8) and non-degeneracy of ω.

6.1.4 Interpretation in terms of open string field theory

The algebraic formulation given above allows us to give an alternate description of the effective superpotential, which makes contact with open string field theory as formulated by Zwiebach (see [34] and references therein). Let us consider the object:

$$\psi := \sum_a s_a \psi_a\,, \tag{6.21}$$

viewed as an element of the graded vector space $H^e_o := \mathbb{A} \otimes H_o$, which is naturally a super bi-module over the commutative superalgebra $\mathbb{A} = \pi(\hat{\mathbb{A}}) \subset \mathcal{B}$ (the notation here follows Subsection 4.2). When H_o is endowed with the suspended degree deg, the quantity ψ has even degree as an element of this module. Using definition (6.15), we find the following expression for the deformed boundary amplitudes:

$$\mathcal{F}_{a_0\dots a_m}(t) = \omega(\psi_{a_0}, r^t_m(\psi_{a_1}\dots\psi_{a_m}))\,.$$

We would like to express this in terms of ψ. For this, consider the natural extension of ω to H^e_o, which we shall denote by the same letter. This is an $\mathbb{A}$-valued bilinear form on H^e_o given as follows on decomposable elements:

$$\omega(f\otimes\psi_a, g\otimes\psi_b) = (-1)^{\tilde{a}\tilde{g}} fg\omega_{ab}\,, \tag{6.23}$$

where f, g are homogeneous elements of $\mathbb{A}$ of degrees $\tilde{f}$ and $\tilde{g}$. We also extend r^t_m to multilinear products on H^e_o (again denoted by the same symbol) through:

$$r^t_m(f_1\psi_{a_1}\dots f_m\psi_{a_m}) = (-1)^{\sum_{j=2}^m(\tilde{a}_1+\dots+\tilde{a}_{j-1})\tilde{f}_j + \tilde{f}_1+\dots+\tilde{f}_m} f_1\dots f_m r^t_m(\psi_{a_1}\dots\psi_{a_m})\,.$$

With these definitions, we have:

$$r^t_m(\psi\dots\psi) = s_{a_m}\dots s_{a_1} r^t_m(\psi_{a_1}\dots\psi_{a_m})$$

and:

$$s_{a_m}\dots s_{a_0}\mathcal{F}_{a_0\dots a_m}(t) = \omega(\psi, r_m(\psi\dots\psi))\,.$$

Thus equation (4.12) becomes:

$$\mathcal{W}(s,t) = \sum_{m\geq 0}\frac{1}{m+1}\omega(\psi, r^t_m(\psi^{\otimes m}))\,.$$

This is the standard form [29] of an open string field action, though built around a background which need not satisfy the open string equations of motion (as reflected by the presence of the product r_0^t). In this interpretation, the object ψ is identified with the string field. As expected, the parameters t encode a deformation of this action, parameterized by a choice of the closed string background. The fact that the effective superpotential of a topological open string theory can be viewed as a string field action follows from the observation that the renormalization group flow[9] in the "target space" (=string field) formulation of such models is a semigroup of homotopy equivalences — thus no information is lost when passing from the microscopic to the long wavelength description.

Fixing the closed string background (i.e. treating t_i as fixed parameters), the open string equations of motion take the form:

$$(\partial_a \hat{\mathcal{W}})(s,t) = 0 \qquad \text{for all } a \iff \sum_{m=0}^{\infty} r_m^t(\psi^{\otimes m}) = 0\,, \tag{6.28}$$

where we assume that $\mathcal{B}$ is chosen such that the extended bilinear form (6.23) is nondegenerate.

This algebraic condition is known as the Maurer-Cartan equation for a weak A_∞ algebra. The presence of r_0^t signals the fact that the reference point $s=0$ does not satisfy this equation. Indeed, the left hand side of (6.28) at $s=0$ equals $r_0(\psi^{\otimes 0}) := r_0(1)$.

6.1.5 Canceling the tadpole

As mentioned above, deformations of the closed string background will generally produce a tadpole which must be canceled if the deformed theory is to have a chance of being conformal. In this subsection, we explain how this can be achieved by shifting the open string vacuum, thereby making contact with previous mathematical work.

Consider a shift:

$$s_a \to s_a + \sigma_a\,, \tag{6.29}$$

with $\sigma_a \in \mathbb{A}$. In terms of the string field (6.21), this operation amounts to:

$$\psi \to \psi + \alpha\,,$$

where $\alpha := \sum_a \sigma_a \psi_a$ is an even element of H_o^e.

It is not hard to check that under such a transformation the deformed scattering products change as:

$$r_m^t \to r_m^{t;\sigma}$$

where:

$$r_m^{t;\sigma}(u_1 \dots u_m) = r^t(e^\alpha, u_1, e^\alpha, u_2, \dots, e^\alpha, u_m, e^\alpha) \tag{6.32}$$

for all $u_1 \dots u_m \in H_o^e$.

[9] The RG flow in open string field theory was studied from the algebraic point of view in [70, 39]. It corresponds to changing the parameter l giving the length of external strips used in the construction of open string vertices in the non-polynomial formulation (see, for example, [34]).

In equation (6.32), we used the notations:

$$e^\alpha := \sum_{k=0}^{\infty} \alpha^{\otimes k}$$

and:

$$r^t := \sum_{m=0}^{\infty} r_m^t \,.$$

Notice that r^t is a map from $\oplus_{m=0}^{\infty}(H_o^e)^{\otimes m}$ to H_o^e.

In particular, the product r_0^t becomes:

$$r_0^{t;\sigma} = r^t(e^\alpha) = \sum_{m=0}^{\infty} r_m^t(\alpha \ldots \alpha)\,.$$

Hence the tadpole amplitude $B_a(t) = \mathcal{F}_a(t) = \omega(\psi_a, r_0^t(1))$ vanishes if and only if:

$$\sum_{m=0}^{\infty} r_m^t(\alpha \ldots \alpha) = 0 \Longleftrightarrow (\partial_a \hat{\mathcal{W}})(\sigma, t) = 0\,. \tag{6.36}$$

This amounts to the well-known fact that the equations of motion for (open) string theory amount to the tadpole cancellation condition. It is not hard to check by direct computation that the products $r_m^{t;\sigma}$ with $m \geq 1$ form a *strong* A_∞ algebra provided that this equation is satisfied. Hence the Maurer-Cartan condition (6.36) describes possible transformations of a weak A_∞ algebra into a (strong) A_∞ algebra obtained by shifts of the form (6.29).

Given a solution σ of (6.36), the expansion of $\mathcal{W}$ around the new open string vacuum takes the form:

$$\mathcal{W}(s,t) = \sum_{m \geq 2} \frac{1}{m+1} \omega(\psi, r_m^{t,\sigma}(\psi^{\otimes m})) + \mathcal{W}(\sigma, t)\,.$$

Up to the last term (which is s-independent), this is the standard form of the open string field action in the formulation of [29].

We mention that condition (6.36) plays a crucial role in the work of [43, 45, 47, 48], where it originates in a very similar manner (see [50] for a detailed discussion).

6.1.6 Relation to the deformation theory of cyclic A_∞ algebras

The results deduced in this subsection are intimately related to the deformation theory of cyclic A_∞ algebras as developed in [71]. This interpretation is quite obvious, so we can be brief.

It is clear from the discussion above that insertion of bulk operators realizes an all-order deformation of the A_∞ algebra of section 5, viewed as a weak A_∞ algebra which happens to be strong and minimal for the particular value $t = 0$ of the deformation parameters. Moreover, such deformations preserve cyclicity and unitality.

To make contact with the work of [71], let us consider the case of infinitesimal deformations discussed in Subsection 6.1.1. This can be recovered from the more general results of the previous subsection by expanding the products r_m^t to first oder in t. Writing:

$$r_m^t = r_m + t_i \Phi_m^i + \mathcal{O}(t^2)\,,$$

we extract morphisms:

$$\Phi^i_m = \frac{\partial r^t_m}{\partial t_i}\Big|_{t=0} : H_o^{\otimes m} \to H_o \,. \tag{6.39}$$

The objects $\Phi^i := \sum_{m=0}^{\infty} \Phi^i_m$ belong to the so-called (weak) Hochschild complex $C = \oplus_{m=0}^{\infty} C^m(H_o)$ of H_o, whose graded subspaces are defined through:

$$C^m(H_o) := Hom(H_o^{\otimes m}, H_o)$$

and whose differential is given by the first order variation of the weak A_∞ constraints (6.17):

$$(\delta\Phi^i)_m(\psi_{a_1}\dots\psi_{a_m}) = (\partial_i A_m)|_{t=0} \,, \tag{6.41}$$

where $A_m(t)$ is the left hand side of (6.17):

$$A_m(t) := \sum_{\substack{k+l=m+1\\ j=0\dots k-1}}^{m} (-1)^{\tilde{a}_1+\dots+\tilde{a}_j} r^t_k(\psi_{a_1}\dots\psi_{a_j}, r^t_l(\psi_{a_{j+1}}\dots\psi_{a_{j+l}}), \psi_{a_{j+l+1}}\dots\psi_{a_m}) \,.$$

In equation (6.41), it is understood that we replace $\frac{\partial r^t_m}{\partial t_i}|_{t=0}$ by Φ^i_m through relation (6.39) and view the result $\delta\Phi^i$ as the action of an algebraic operator[10] δ on Φ^i. The fact that δ squares to zero follows from the A_∞ constraints.

Because our A_∞ algebras are cyclic, one has further restrictions on Φ^i which amount to the statement that they are elements of a certain subcomplex $CC(H_o)$ known as the *cyclic complex*. This can be defined as the set of elements $\Phi = \sum_m \Phi_m$ in $C(H_o)$ with the property that the quantities $\omega(\psi_{a_0}, \Phi_m(\psi_{a_1}\dots\psi_{a_m}))$ obey the cyclicity constraints:

$$\omega(\psi_{a_0}, \Phi_m(\psi_{a_1}\dots\psi_{a_m})) = (-1)^{\tilde{a}_m(\tilde{a}_0+\dots+\tilde{a}_{m-1})}\omega(\psi_{a_m}, \Phi_m(\psi_{a_1}\dots\psi_{a_{m-1}})) \,.$$

For our maps Φ^i, these conditions follow by differentiating (6.18) with respect to t_i at $t=0$. The Hochschild differential δ preserves the subspace $CC(H_o)$. Denoting its restriction by the same letter, one obtains the cyclic complex $(CC(H_o), \delta)$ considered[11] in [71].

Since the deformed products (6.15) obey weak A_∞ constraints for all t, differentiation of (6.17) at $t=0$ shows that Φ^i are δ-closed:

$$\delta\Phi^i = 0 \,.$$

Thus Φ^i define elements $[\Phi^i]$ of the cohomology of $(CC(H_o), \delta)$, known as the (weak) cyclic cohomology of the A_∞ algebra (H_o, r_*). Comparing with Subsection 6.1.1, it is easy to see that Φ^i can be written as:

$$\Phi^i_m(\psi_{a_1}\dots\psi_{a_m}) = B^a_{a_1\dots a_m;i}\psi_a \,.$$

[10] Strictly speaking, this specifies δ only for elements Φ of $C(H_o)$ such that each Φ_m has degree one as a map from $H_o^{\otimes m}$ to H_o. However the definition generalizes to the entire Hochschild complex.

[11] Our sign conventions differ from those of [71] by suspension. Moreover, we allow for the subspace $C^0(H_0) = CC(H_0) = \mathbb{C}$ in the Hochschild and cyclic complexes, since we consider deformations of *weak* and cyclic A_∞ algebras.

This shows that they are completely determined by the disk amplitudes $B_{a_0\dots a_m;i}$ with a single bulk insertion. Thus one has a map:

$$\phi_i \to [\Phi^i]$$

from BRST-closed bulk zero-form observables to the cyclic cohomology of the A_∞ algebra (H_o, r_*). A similar statement was proposed in [26] in a particular case.

6.2 Bulk-boundary crossing symmetry

The second bulk-boundary crossing constraint (2.32) of two-dimensional topological field theory states that the bulk-boundary map is a morphism from the bulk to the boundary algebra [33]. In this section, we discuss the 'stringy' generalization of this constraint.

It is clear that the relevant property arises from factorization of the amplitude:

$$\left\langle \phi_i \phi_j \, \psi_{a_0} P \int \psi_{a_1}^{(1)} \cdots \int \psi_{a_m}^{(1)} \right\rangle, \tag{6.47}$$

into the channel where the two bulk fields approach each other and the channel where both bulk fields approach the boundary. In contrast to the A_∞ constraints, this factorization follows from explicit movement of the bulk operators rather than from the Ward identities of the BRST symmetry. This is similar to the mechanism leading to the WDVV equations (2.13). In the case at hand we have to deal with a subtlety which requires closer examination: we know from section 3 that only the fundamental amplitudes (3.12) are independent of the positions of unintegrated insertions. This is not the case for the amplitude (6.47), since it contains two bulk and one boundary unintegrated insertions. Therefore, it is not immediately clear that factorizing (6.47) makes sense. The naive guess for the factorization takes the form:

$$\begin{aligned}
&C_{ij}{}^{l} \left\langle \phi_l \, \psi_{a_0} P \int \psi_{a_1}^{(1)} \cdots \int \psi_{a_m}^{(1)} \right\rangle_{D^2} = \\
&\quad = \sum_{0 \le m_1 \le \cdots m_4 \le m} \left\langle \psi_{a_0} P \int \psi_{a_1}^{(1)} \cdots \int \psi_{a_{m_1}}^{(1)} \, \psi_b \, P \int \psi_{a_{m_2+1}}^{(1)} \times \cdots \right. \\
&\qquad\qquad \left. \times \int \psi_{a_{m_3}}^{(1)} \, \psi_c \, P \int \psi_{a_{m_4+1}}^{(1)} \cdots \int \psi_{a_m}^{(1)} \right\rangle_{D^2} \times \\
&\quad \times \omega^{bd} \, \omega^{ce} \left\langle \phi_i \, \psi_d P \int \psi_{a_{m_1+1}}^{(1)} \cdots \int \psi_{a_{m_2}}^{(1)} \right\rangle_{D^2} \left\langle \phi_j \; \psi_e P \int \psi_{a_{m_3+1}}^{(1)} \cdots \int \psi_{a_{m_4}}^{(1)} \right\rangle_{D^2} .
\end{aligned} \tag{6.48}$$

Since the correlation function (6.47) is not independent of the positions of the fixed insertions, we shall give an independent argument for why this relation holds. In the following, we shall denote the left hand side of (6.48) simply by (l.h.s.), and the right hand side by (r.h.s.).

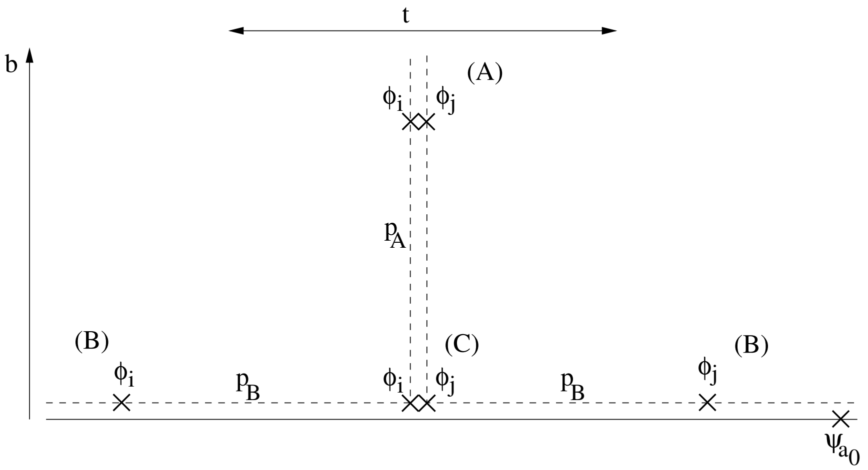

Figure 5: The factorization associated with the stringy version of the second bulk-boundary crossing constraint. Configuration (A) corresponds to the topological bulk product and (B) to the factorization at the boundary. Configuration (C) connects these channels. The quantity b is the equal distance of the bulk fields from the boundary, while t is the horizontal separation of the bulk fields.

To establish equation (6.48), we consider the amplitude (6.47) for the configurations (A) and (B) of bulk operators on the upper half plane shown in figure 5. Let us denote the distance between the bulk operators by $t \in \mathbb{R}$ and assume that the two bulk operators sit on a line parallel at a distance b to the boundary. In the limit $t \to 0$, configuration (A) corresponds to the left hand side of equation (6.47), while the right hand side of this equation arises in the limit $b \to 0$ of configuration (B).

For configuration (A), we have $t = t_0$ with $|t_0| \ll 1$ and we can perform a bulk operator product expansion in t_0, so that (6.47) becomes (l.h.s.) $+ g_1(t_0, b)$, where $g_1(t_0, b) = \mathcal{O}(t_0)$ denotes contributions from higher terms in the OPE. Moving along the path p_A down toward configuration (C), the function $g_1(t_0, b)$ changes with b and becomes $g_1(t_0, b_0)$. Configuration (B) shows the bulk operators at the distance $b = b_0 \ll 1$ from the boundary. According to the bulk-boundary operator product the amplitude (6.47) takes the form (r.h.s.) $+ g_2(t, b_0)$, where $g_2(t, b_0) = \mathcal{O}(b_0)$. Following the path p_B we reach again the point (C). At (C) we have (l.h.s.) $+ g_1(t_0, b_0) =$ (r.h.s.) $+ g_2(t_0, b_0)$, which implies that $g_1(t_0, b_0)$ and $g_2(t_0, b_0)$ are non-singular for $b_0 \to 0$ and $t_0 \to 0$, respectively. Hence we can safely take the factorization limit $t_0, b_0 \to 0$, in which g_1 and g_2 vanish, so that (l.h.s.) = (r.h.s.). This shows that equation (6.48) holds.

Using the Ward identity (3.33) to move the integral contours, and taking into account definition (3.12), equation (6.48) gives:

$$B^{a_0}{}_{a_1 \ldots a_m ; l} C^l{}_{ij} = \sum_{0 \le m_1 \le \ldots m_4 \le m} (-1)^{\tilde{a}_{m_1+1} + \cdots + \tilde{a}_{m_3}} B^{a_0}{}_{a_1 \ldots a_{m_1} b a_{m_2+1} \ldots a_{m_3} c a_{m_4+1} \ldots a_m} \times \\ \times B^b{}_{a_{m_1+1} \ldots a_{m_2} ; i} B^c{}_{a_{m_3+1} \ldots a_{m_4} ; j} \, , \tag{6.49}$$

where $C^l{}_{ij}$ are the usual bulk ring structure constants. Note that the left-hand side is manifestly symmetric in i and j whereas this symmetry is not manifest in the right-hand side. This reflects the fact that one can also accomplish the factorization of figure 5 after exchanging i and j.

Additional integrated bulk insertions spread in the usual way when we factorize, so we can treat them as derivatives and combine all relations into a single equation involving the quantities $\mathcal{F}_{a_0\ldots a_m}(t)$ for $m \geq 0$ and the bulk WDVV potential $\mathcal{F}(t)$:

$$\begin{aligned}\partial_i\partial_j\partial_k\mathcal{F}(t)\,\eta^{kl}\,\partial_l\mathcal{F}_{a_0a_1\ldots a_m}(t) = \sum_{0\leq m_1\leq\ldots m_4\leq m} & (-1)^{\tilde{a}_{m_1+1}+\cdots+\tilde{a}_{m_3}}\times\\ & \times\mathcal{F}_{a_0\ldots a_{m_1}ba_{m_2+1}\ldots a_{m_3}ca_{m_4+1}\ldots a_m}(t)\times\\ & \times\partial_i\mathcal{F}^b{}_{a_{m_1+1}\ldots a_{m_2}}(t)\,\partial_j\mathcal{F}^c{}_{a_{m_3+1}\ldots a_{m_4}}(t)\,.\end{aligned}$$

For $m = 0$ and $m = 1$, these equations take the form:

$$\partial_i\partial_j\partial_k\mathcal{F}\,\eta^{kl}\,\partial_l\mathcal{F}_{a_0} = \mathcal{F}_{a_0bc}\,\partial_i\mathcal{F}^b\,\partial_j\mathcal{F}^c\,, \tag{6.50}$$

$$\begin{aligned}\partial_i\partial_j\partial_k\mathcal{F}\;\eta^{kl}\;\partial_l\mathcal{F}_{a_0a_1} = {} & \mathcal{F}_{a_0bca_1}\;\partial_i\mathcal{F}^b\;\partial_j\mathcal{F}^c+\mathcal{F}_{a_0bc}\;\partial_i\mathcal{F}^b\;\partial_j\mathcal{F}^c{}_{a_1}+\\ & +(-1)^{\tilde{a}_1}\;\mathcal{F}_{a_0ba_1c}\;\partial_i\mathcal{F}^b\;\partial_j\mathcal{F}^c+(-1)^{\tilde{a}_1}\;\mathcal{F}_{a_0bc}\;\partial_i\mathcal{F}^b{}_{a_1}\;\partial_j\mathcal{F}^c+\\ & +\mathcal{F}_{a_0a_1bc}\;\partial_i\mathcal{F}^b\;\partial_j\mathcal{F}^c\,.\end{aligned} \tag{6.51}$$

The undeformed version of (6.50) coincides with the second bulk-boundary crossing constraint (3.19) of two-dimensional TFT.

6.3 Cardy conditions

The Cardy condition is probably the most interesting sewing constraint of 2d TFT [31, 32, 33], since it connects the exchange of closed strings between D-branes at the tree level with a one-loop open string amplitude. Allowing for insertions of both bulk and boundary fields in the corresponding cylinder amplitude leads to the following factorization:

$$\begin{aligned}&\left\langle\phi_i\psi_{a_0}P\int\psi_{a_1}\cdots\int\psi_{a_n}\right\rangle\eta^{ij}\left\langle\phi_j\psi_{b_0}P\int\psi_{b_1}\cdots\int\psi_{b_m}\right\rangle=\\ &=\sum_{\substack{0\leq n_1\leq n_2\leq n\\0\leq m_1\leq m_2\leq m}}(-1)^s\,\omega^{c_1d_1}\omega^{c_2d_2}\left\langle\psi_{a_0}\;P\int\psi_{a_1}\cdots\int\psi_{a_{n_1}}\psi_{d_1}P\int\psi_{b_{m_1+1}}\times\cdots\right.\\ &\qquad\left.\times\int\psi_{b_{m_2}}\psi_{c_2}P\int\psi_{a_{n_2+1}}\cdots\int\psi_{a_n}\right\rangle\times\\ &\qquad\times\left\langle\psi_{b_0}\;P\int\psi_{b_1}\cdots\int\psi_{b_{m_1}}\psi_{c_1}P\int\psi_{a_{n_1+1}}\times\cdots\right.\\ &\qquad\left.\times\int\psi_{a_{n_2}}\psi_{d_2}P\int\psi_{b_{m_2+1}}\cdots\int\psi_{b_m}\right\rangle,\end{aligned} \tag{6.52}$$

where the sign s accounts for reshuffling of the boundary fields. The left hand side of (6.52) is the factorization in the closed string channel, in which the cylinder becomes infinitely long. The right hand side corresponds to the generalization of the double-twist diagram [32] of the open string channel.

Taking into account further integrated bulk insertions, equations (6.52) become:

$$\partial_i \mathcal{F}_{a_0\ldots a_n}\eta^{ij}\partial_j\mathcal{F}_{b_0\ldots b_m} = \sum_{\substack{0\le n_1\le n_2\le n\\ 0\le m_1\le m_2\le m}} (-1)^{s+\tilde{c}_1+\tilde{c}_2}\ \omega^{c_1d_1}\ \omega^{c_2d_2}\ \times \\ \times\, \mathcal{F}_{a_0\ldots a_{n_1}d_1 b_{m_1+1}\ldots b_{m_2}c_2 a_{n_2+1}\ldots a_n} \times \\ \times\, \mathcal{F}_{b_0\ldots b_{m_1}c_1 a_{n_1+1}\ldots a_{n_2}d_2 b_{m_2+1}\ldots b_m}\,. \tag{6.53}$$

The first relations in this hierarchy of constraints take the form:

$$\begin{aligned} \partial_i\mathcal{F}_{a_0}\eta^{ij}\partial_j\mathcal{F}_{b_0} &= (-1)^{s+\tilde{c}_1+\tilde{c}_2}\ \omega^{c_1d_1}\ \omega^{c_2d_2}\ \mathcal{F}_{a_0d_1c_2}\ \mathcal{F}_{b_0c_1d_2}\,,\\ \partial_i\mathcal{F}_{a_0a_1}\eta^{ij}\partial_j\mathcal{F}_{b_0} &= (-1)^{s+\tilde{c}_1+\tilde{c}_2}\ \omega^{c_1d_1}\ \omega^{c_2d_2}\ \mathcal{F}_{a_0d_1c_2a_1}\ \mathcal{F}_{b_0c_1d_2} + \\ &\quad +(-1)^{s+\tilde{c}_1+\tilde{c}_2}\ \omega^{c_1d_1}\ \omega^{c_2d_2}\ \mathcal{F}_{a_0d_1c_2}\ \mathcal{F}_{b_0c_1a_1d_2} + \\ &\quad +(-1)^{s+\tilde{c}_1+\tilde{c}_2}\ \omega^{c_1d_1}\ \omega^{c_2d_2}\ \mathcal{F}_{a_0a_1d_1c_2}\ \mathcal{F}_{b_0c_1d_2}\,. \end{aligned} \tag{6.54}$$

Taking the limit $t = 0$ in the first equation recovers the Cardy constraint (3.20) of two-dimensional TFT. Notice that the left hand side of the first equation vanishes identically if we consider insertions of the identity operator, and if the suspended degree $\tilde{\omega}$ of the symplectic structure vanishes; this reflects vanishing of the Witten index in that case.

It is worth pointing out that the arguments of section 3 cannot be used to show that the annulus amplitude is independent on the world-sheet metric and of the positions of unintegrated boundary insertions. In fact, experience with the bulk theory [5] suggests that there are BRST anomalies in open string correlators beyond tree level, so there is indeed no *a priori* reason why the annulus amplitude should be metric-independent. However, we will take the point of view that when *imposing* the Cardy condition (6.53), one focuses by definition on the topological part of the amplitude. It is not clear to us whether the Cardy relation is satisfied by the complete amplitude, which potentially involves supplementary anomalous contributions.

In the present paper we will be concerned only with the topological part of the annulus amplitude.[12] To capture the full amplitude including possible holomorphic anomalies would require the analog of $t - t^*$ equations [5] for open strings, a subject which is beyond the scope of the present paper.

7. Application: superpotentials for D-branes in topological minimal models

In this section, we demonstrate the power of the consistency conditions derived in this paper (namely cyclicity (3.35)), weak A_∞ structure (6.14), bulk-boundary sewing (6.50) and Cardy relations (6.53)) by applying them to certain families of D-branes in B-type topological minimal models. In the examples considered below, we shall find that the totality of these constraints suffices to determine the effective superpotential.

[12]For the Landau-Ginzburg examples described in section 4, we shall find that imposing the generalized Cardy condition as written above agrees with independently known results (namely, with the factorization property of the Landau-Ginzburg potential).

Let us recall some facts [57]–[60] about D-branes in B-type topological minimal models. As usual, the bulk sector is characterized by the level k, while D-brane boundary sectors are labeled by $\kappa = 0, 1, \ldots [k/2]$. It is convenient to switch to the Landau-Ginzburg realization of these models. Then the bulk sector is described by a univariate polynomial $W_{LG}^{(k+2)}(\varphi)$ of degree $k+2$ in the complex variable φ, which gives the worldsheet superpotential. On the other hand, 'non-multiple' B-type D-branes in the boundary sector κ correspond to factorizations of the bulk superpotential:

$$W_{LG}^{(k+2)}(\varphi) = J^{(\kappa+1)}(\varphi)\, E^{(k+1-\kappa)}(\varphi)\,, \qquad \kappa = 0, 1, \ldots \left[\frac{k}{2}\right] \tag{7.1}$$

where $J^{(\kappa+1)}(\varphi)$ is a polynomial of degree $\kappa+1$ [57, 58].

The open string spectrum consists of boundary changing and boundary preserving sectors. We focus first on boundary preserving sectors, each of which corresponds to a degree label κ. The on-shell boundary algebra H_o is isomorphic with a supercommutative ring $\mathcal{R}_\partial$ with even and odd generators φ and ω, subject to relations which can be described as follows. Let H denote the greatest common denominator of J and E, ie., $J^{(\kappa+1)}(\varphi) = p(\varphi)H^{(\ell+1)}(\varphi)$ and $E^{(k+1-\kappa)}(\varphi) = q(\varphi)H^{(\ell+1)}(\varphi)$ for some polynomials p, q.[13] Then the relations in the boundary ring are [58]:

$$\mathcal{I}:\ \left\{H^{(\ell+1)}(\varphi) = 0\,,\quad \omega^2 \ = \ p(\varphi)q(\varphi)\right\}. \tag{7.2}$$

The U(1) charges of the generators are given by:

$$q(\phi) = 1\,, \qquad q(\omega) = \frac{k}{2} - \ell\,.$$

When viewed as a complex vector space, the boundary algebra H_o admits the basis:

$$\mathcal{R}_\partial \equiv \ \{\,\psi_a\,\} = \{\varphi^\alpha, \omega\varphi^\alpha\}\,, \qquad a = 0, \ldots, 2\ell+1\,, \quad \alpha = 0, \ldots, \ell\,. \tag{7.3}$$

Recall that the bulk algebra is given by the Newton ring $\mathcal{R} = \mathbb{C}[\phi]/\langle \partial_\varphi W_{LG}^{(k+2)}(\varphi)\rangle$, which admits the following basis when viewed as a complex vector space:

$$\mathcal{R} \equiv \{\,\phi_i\,\} = \{\,\varphi^i\,\}\,, \qquad i = 0, \ldots, k\,.$$

When suitably integrated, each of the fields can be used to deform the theory. In the bulk sector we have the deformation $\delta S = \sum_{i=0}^{k} t_{k+2-i} \int d^2z [G_{-1}, [\bar{G}_{-1}, \phi_i]$, while in the boundary sector we have:

$$\delta S_\partial = \left(\sum_{\alpha=0}^{\ell} u_{\ell+1-\alpha} \int dx\, G(\omega\varphi^\alpha)\right) + \left(\sum_{\alpha=0}^{\ell} v_{k/2+1-\alpha} \int dx\, G\varphi^\alpha\right). \tag{7.4}$$

In this equation, we divided the boundary deformation parameters s_a into even and odd variables u_α and v_α. These parameters can be formally assigned U(1) charges, which can

[13]In the unperturbed theory, about which we will expand, we can take both p and q to be constant, and we normalize them by setting $pq = -\frac{\varphi^{k-2\kappa}}{k+2}$.

be used as labels; this is the convention employed in equation (7.4). Notice that super-integration over the moduli of boundary punctures flips the $\mathbb{Z}_2$ degree, so that odd ring elements (=topological tachyon excitations $\omega\phi^\alpha$) are associated with the bosonic deformation parameters u, and vice versa.

We are now ready to present some computations. We first consider a few explicit examples and determine their effective superpotentials. As a rule, we shall find that the generalized WDVV equations lead to a unique solution, but only once *all* constraints are imposed on the open-closed amplitudes. For example, fixing $t = 0$ and imposing only the A_∞ conditions (5.4) leaves some parameters undetermined in the effective superpotential $\mathcal{W}(0, s) = \mathcal{W}(0, u, v)$. It is only after considering both open and closed deformations and imposing the bulk-boundary constraints (6.50) and Cardy conditions (6.53) that all coefficients of $\mathcal{W}(s, t)$ become uniquely determined.[14]

For the example $(k, \ell) = (3, 1)$, we find the following expressions for the perturbed boundary correlators (4.4) on the disk:

$$\begin{aligned}
\mathcal{F}_{021} = -\mathcal{F}_{003} = -\mathcal{F}_{012} = -\mathcal{F}_{1213} &= 1\,, \\
\mathcal{F}_{222} = \mathcal{F}_{2233} = \mathcal{F}_{2323} = \mathcal{F}_{23333} = \mathcal{F}_{333333} &= -\frac{1}{5}\,, \\
\mathcal{F}_{22} = \mathcal{F}_{233} = \mathcal{F}_{3333} &= t_2\,, \\
\mathcal{F}_{23} = \mathcal{F}_{333} &= t_3\,, \\
\mathcal{F}_2 = \mathcal{F}_{33} &= t_4 - {t_2}^2, \\
\mathcal{F}_3 &= t_5 - t_2\, t_3\,.
\end{aligned} \tag{7.5}$$

Our notation was explained after equations (7.3), namely $a = 0, 1$ and $a = 2, 3$ label even and odd boundary ring elements respectively. Moreover, we listed one representative per cyclic orbit, and only the non-vanishing amplitudes. The value of $-1/5$ for the correlators which contain three unintegrated fermionic insertions arises from our normalization, which is $\omega^2 = -\varphi/5$. Notice that ordering of boundary indices is indeed important; for example $\mathcal{F}_{1123} = 0$ while $\mathcal{F}_{1213} = -1$.

In this example, the effective superpotential takes the form:

$$\begin{aligned}
-\mathcal{W}(t, u) = &\frac{1}{5}\left(\frac{{u_1}^6}{6} + {u_1}^4 u_2 + \frac{3}{2}{u_1}^2{u_2}^2 + \frac{{u_2}^3}{3}\right) + t_2\left(\frac{-{u_1}^4}{4} - {u_1}^2 u_2 - \frac{{u_2}^2}{2}\right) - \\
&-t_3\left(\frac{{u_1}^3}{3} + u_1 u_2\right) + \left(t_4 - {t_2}^2\right)\left(\frac{-{u_1}^2}{2} - u_2\right) - \left(t_5 - t_2 t_3\right) u_1\,.
\end{aligned} \tag{7.6}$$

Since the parameters v are odd while appearing only in anti-commutators, they drop out from the effective superpotential, even though the corresponding non-symmetrized amplitudes are non-zero and have to be taken into account when solving the constraint equations.

[14] Strictly speaking, this is true only up to choosing the normalization of the 3-point boundary and 2-point bulk-boundary correlators. In the computations below, we normalized these correlators in a manner which is natural in the LG description. Notice that the sign in the Cardy condition (6.53) is given by: $(-1)^s = (-1)^{(\tilde{c}_1+\tilde{a}_0)(\tilde{c}_2+\tilde{b}_0)}$ in the present case.

The effective superpotentials for some other examples are as follows. For $(k,\ell)=(4,2)$ we find:

$$
\begin{aligned}
-\mathcal{W}(t,u) \;=\; & \frac{1}{6}\left(\frac{{u_1}^7}{7}+{u_1}^5u_2+2{u_1}^3{u_2}^2+u_1{u_2}^3+{u_1}^4u_3+3{u_1}^2u_2u_3+{u_2}^2u_3+u_1{u_3}^2\right)-\\
& -t_2\left(\frac{{u_1}^5}{5}+{u_1}^3u_2+u_1{u_2}^2+{u_1}^2u_3+u_2u_3\right)+\\
& +t_3\left(\frac{-{u_1}^4}{4}-{u_1}^2u_2-\frac{{u_2}^2}{2}-u_1u_3\right)+\left(\frac{t_4-3{t_2}^2}{2}\right)\left(\frac{-{u_1}^3}{3}-u_1u_2-u_3\right)+\\
& +\Big(t_5-2t_2t_3\Big)\left(\frac{-{u_1}^2}{2}-u_2\right)-\left(t_6+\frac{{t_2}^3}{3}-\frac{{t_3}^2}{2}-t_2t_4\right)u_1\,,
\end{aligned}
$$

while for $(k,\ell)=(5,2)$ we obtain:

$$
\begin{aligned}
-\mathcal{W}(t,u) \;=\; & \frac{1}{7}\left(\frac{{u_1}^8}{8}+{u_1}^6u_2+\frac{5{u_1}^4{u_2}^2}{2}+2{u_1}^2{u_2}^3+\frac{{u_2}^4}{4}+{u_1}^5u_3+\right.\\
& \left.\quad+4{u_1}^3u_2u_3+3u_1{u_2}^2u_3+\frac{3{u_1}^2{u_3}^2}{2}+u_2{u_3}^2\right)-\\
& -t_2\left(\frac{{u_1}^6}{6}+{u_1}^4u_2+\frac{3{u_1}^2{u_2}^2}{2}+\frac{{u_2}^3}{3}+{u_1}^3u_3+2u_1u_2u_3+\frac{{u_3}^2}{2}\right)+\\
& +t_3\left(\frac{-{u_1}^5}{5}-{u_1}^3u_2-u_1{u_2}^2-{u_1}^2u_3-u_2u_3\right)+\\
& +\big(t_4-2{t_2}^2\big)\left(\frac{-{u_1}^4}{4}-{u_1}^2u_2-\frac{{u_2}^2}{2}-u_1u_3\right)+\\
& +\big(t_5-3t_2t_3\big)\left(\frac{-{u_1}^3}{3}-u_1u_2-u_3\right)+\big(t_6+{t_2}^3-{t_3}^2-2t_2t_4\big)\left(\frac{-{u_1}^2}{2}-u_2\right)-\\
& -\big(t_7+{t_2}^2t_3-t_3t_4-t_2t_5\big)u_1\,.
\end{aligned}
$$

Notice that $\mathcal{W}(t,u)$ has U(1) charge equal to $k+3$, which is one-half of the charge of the effective prepotential $\mathcal{F}(t)$ of the bulk sector.

These results, obtained by painstakingly solving the generalized WDVV constraints, suggest the following closed formula for the effective superpotential in the general boundary preserving sector labeled by (k,ℓ):[15]

$$\mathcal{W}(t,u)=-\sum_{i=0}^{k+2} g^{(k)}_{k+2-i}(t)\,h^{(\ell)}_{i+1}(u)\,, \tag{7.7}$$

where $h^{(\ell)}_i(u)$ are defined by:

$$\log\left[1-\sum_{n=1}^{\ell+1}u_n\,y^n\right]:=\sum_{i=1}^{\infty}h^{(\ell)}_i(u)\,y^i \tag{7.8}$$

and $g^{(k)}_{k+2-i}(t)$ are the coefficients of φ^i in the bulk LG superpotential:

$$W^{(k+2)}_{LG}(t)=-\sum_{i=0}^{k+2}g^{(k)}_{k+2-i}(t)\,\varphi^i\,,$$

[15]We plan to discus this in more detail elsewhere.

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whose explicit form can be found for example in [12] (here $g_0^{(k)} = -1/(k+2)$ and $g_1^{(k)} = 0$). Equation (7.7) implies the following expression for the deformed one-point correlators on the disk:

$$\mathcal{F}_{\ell+1-\alpha}(t) \equiv \partial_{u_\alpha} \mathcal{W}(t,u)|_{u=0} = g_{k+3-\alpha}(t)\,.$$

As explained in Subsection 6.1.5, in the presence of deformations the tadpoles must be canceled by shifting to a new vacuum for which $\partial_{u_\alpha}\mathcal{W}(t,u) = 0$. A reassuring consistency check is provided by solving this condition, where t_i are treated as parameters. These equations give a set of constraints relating u and t, thereby determining an affine algebraic variety $\mathcal{Z}_{k,l}$. This can be parameterized by solving for $\{t_\bullet\} = \{t_{k+2-\ell}, \ldots, t_{k+2}\}$ in terms of u_α and $\{t_\circ\} = \{t_2, \ldots, t_{k+1-\ell}\}$. We find that the locus $\mathcal{Z}_{k,l}$ has the property that the bulk superpotential factorizes along it as follows:

$$W_{LG}^{(k+2)}(\varphi, u, t_\circ, t_\bullet(t_\circ, u)) = J^{(\ell+1)}(\varphi, u)\, E^{(k+1-\ell)}(\varphi, u, t_\circ)\,, \tag{7.9}$$

with

$$J^{(\ell+1)}(\varphi, u) = \varphi^{\ell+1} - \sum_{\alpha=0}^{\ell} u_{\ell+1-\alpha}\, \varphi^\alpha\,, \tag{7.10}$$

and

$$E^{(k+1-\ell)}(\varphi, u, t_\circ) = -\sum_{i=\ell+1}^{k+2} g_{k+2-i}(t_\circ) \left(\sum_{n=0}^{i-\ell-1} \varphi^n\, f^{(\ell)}_{i-\ell-n-1}(u) \right), \tag{7.11}$$

where the coefficients $f_i^{(\ell)}$ are determined by the relation $\frac{1}{1-\sum_{n=1}^{l+1} u_n y^n} := -\sum_{i=0}^{\infty} f_i^{(l)} y^i$.

In the untwisted model, the physical interpretation is as follows. Generic bulk (t) and boundary (u) perturbations break supersymmetry, a phenomenon which can be traced back to the boundary terms (2.36) and (2.40) in the BRST variation of integrated descendants. Thus t and u 'feel' a potential which represents an obstruction against such deformations. The effects of the boundary terms cancel and supersymmetry is maintained only when bulk and boundary deformations are locked together through the relation $W_{LG} = J\,E$ – this cancellation was indeed precisely why one had to introduce a boundary potential in the first place [72, 57, 58, 60, 61]. Thus it is no surprise, though a welcome check on our computations, that the critical set $\mathcal{Z}$ of $\mathcal{W}(t,u)$ with respect to the boundary deformation parameters u corresponds to the factorization locus of the worldsheet LG superpotential in the combined, bulk and boundary parameter space. As expected, this is precisely the locus along which the boundary data preserve half of the supersymmetry of the worldsheet action, thereby allowing for a meaningful coupling to B-type D-branes. This is similar in spirit to ref. [73], where, in a different physical context, critical points of effective superpotentials were associated with factorization loci in the target space geometry.

If $E(\varphi, u, t_\circ)$ is generic, its greatest common denominator with $J(\varphi, u)$ is trivial, hence according to (7.2) no physical open string states survive after turning on bulk and boundary deformations by allowing for general t, u. This reflects tachyon condensation of the $D2\bar{D}2$ system [74, 57], leading to the trivial open string vacuum. Only upon appropriately specializing $E(\varphi, u, t_\circ)$ such that it has a non-trivial common factor $H(\varphi, u)$ with $J(\varphi, u)$, does one find that some open string states remain in the physical spectrum. Such sub-loci of the factorization variety $\mathcal{Z}_{k,l}$ correspond to (a topological model of) tachyon condensation with non-trivial endpoint. In this version of tachyon condensation, the open string spectrum gets truncated while moving between different strata of the supersymmetry preserving locus of the effective superpotential. A very similar picture was found in [36] for the case of the open A model close to a large radius point of a Calabi-Yau compactification (see figure 2 of that paper). The topological version of tachyon condensation was discussed in detail in [51, 52, 54, 55, 36, 37] in the context of open string field theory. It also plays a central role in the work of [45, 48].

Finally, let us give an example of effective superpotentials for the boundary changing sector of minimal models. For simplicity we will not turn on bulk deformations. In this situation we can study the formation of D-brane composites in a fixed conformal bulk background. Let us consider the minimal model at level $k = 3$ with the D-brane configuration $\mathscr{B}_{\ell=0} \oplus \mathscr{B}_{\ell=1}$. In this setting, one has fermionic boundary operators, $(\omega^{(00)}, \omega^{(01)}, \omega^{(10)}, \omega^{(11)}, \varphi\omega^{(11)})$, associated with the deformation parameters $(u_1^{(00)}, u_{3/2}^{(01)}, u_{3/2}^{(10)}, u_2^{(11)}, u_1^{(11)})$ (see [58]). The generalized WDVV equations again determine all amplitudes, giving the following effective superpotential for the bosonic deformation parameters:

$$\begin{aligned} \mathcal{W}(t=0,u) = & -\frac{1}{30}u_1^{(00)6} - \frac{1}{15}u_2^{(11)3} - \frac{3}{10}u_2^{(11)2}u_1^{(11)2} - \frac{1}{5}u_2^{(11)}u_1^{(11)4} - \frac{1}{30}u_1^{(11)6} - \\ & -\frac{1}{5}u_1^{(00)3}u_{3/2}^{(01)}u_{3/2}^{(10)} - \frac{1}{5}u_1^{(00)}u_2^{(11)}u_{3/2}^{(01)}u_{3/2}^{(10)} - \\ & -\frac{1}{5}u_1^{(00)2}u_1^{(11)}u_{3/2}^{(01)}u_{3/2}^{(10)} - \frac{2}{5}u_2^{(11)}u_1^{(11)}u_{3/2}^{(01)}u_{3/2}^{(10)} - \\ & -\frac{1}{5}u_1^{(00)}u_1^{(11)2}u_{3/2}^{(01)}u_{3/2}^{(10)} - \frac{1}{5}u_1^{(11)3}u_{3/2}^{(01)}u_{3/2}^{(10)} - \frac{1}{10}u_{3/2}^{(01)2}u_{3/2}^{(10)2}\,. \end{aligned} \tag{7.12}$$

In view of the results of this paper, one expects the situation to be similar to that found for boundary preserving sectors, namely the critical set of $\mathcal{W}(t{=}0,u)$ should parameterize deformations compatible with a factorized bulk potential: $W_{LG}^{(k+2)}\mathbb{1} = J^{(01)}E^{(01)} = E^{(01)}J^{(01)}$, where $J^{(01)}$ and $E^{(01)}$ are now matrices comprising both the boundary changing and boundary preserving sectors. Making an appropriate ansatz for the dependence of $J^{(01)}$ and $E^{(01)}$ of the parameters $u_a^{(\ell\ell')}$ (namely an ansatz compatible with the U(1) charges), one indeed finds that the critical locus of $\mathcal{W}(t{=}0,u)$, characterized by:

$$\begin{aligned} u_1^{(11)} &= -u_1^{(00)} \\ u_2^{(11)} &= -u_1^{(00)2} \\ u_{3/2}^{(01)}u_{3/2}^{(10)} &= -u_1^{(00)3}\,, \end{aligned} \tag{7.13}$$

implies factorization of ${W_{LG}}^{(k+2)}\mathbb{1}$. The matrices $J^{(01)}$ and $E^{(01)}$ are uniquely determined by (7.13) and take the form:

$$J^{(01)} = \begin{pmatrix} \varphi - u_1^{(00)} & -u_{3/2}^{(01)} \\ -u_{3/2}^{(10)} & \varphi^2 + u_1^{(00)}\varphi + u_1^{(00)2} \end{pmatrix},$$

$$E^{(01)} = \frac{1}{5}\begin{pmatrix} \varphi^4 + u_1^{(00)}\varphi^3 + u_1^{(00)2}\varphi^2 & u_{3/2}^{(01)}\varphi^2 \\ u_{3/2}^{(10)}\varphi^2 & \varphi^3 - u_1^{(00)}\varphi^2 \end{pmatrix}. \tag{7.14}$$

Although the deformations (7.13) preserve half of bulk supersymmetry, the open string spectrum is truncated for generic values of the boundary deformation parameters. This is different from the boundary flows studied recently in [75], for which the open string spectrum does not truncate along the deformation locus considered there.

8. Outlook

Our work brings up a number of interesting questions. One is how to make contact with the Landau-Ginzburg formulation of B-type topological open strings. This would amount to investigating the A_∞ structure of the contact terms through the methods of open string field theory, as proposed in a general context in [35]. We intend to present our findings in this direction in a subsequent paper.

Another important problem is to apply the generalized WDVV equations to theories allowing for exactly marginal bulk deformations, like Calabi-Yau manifolds with D-branes, and use them to learn about D-brane superpotentials. This was one of the main motivations for the present paper, and should eventually allow one to make contact with geometric computations based on mirror symmetry, such as those performed in [20, 21, 25].

Just as for the bulk theory, one expects that there is a connection of open-closed topological minimal models with matrix models and integrable systems. It would be especially interesting to understand the relation with the recent work of [76]. A related question concerns open string gravitational descendants in these models, which requires a systematic study of topological gravity on bordered Riemann surfaces. In particular, open string gravitational descendants should lead lead to interesting generalizations of the Virasoro and W-constraints. Some work in this direction, though from a different perspective, was recently carried out in [77].

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